Chapter 9
Global Dynamics of Generic 3-Flows

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The results in Chap. 5 form the basis of a theory of flows on three-dimensional manifolds and paved the way for a global understanding of the dynamics of $C^1$ generic flows in dimension 3. Here we present some results from the generic viewpoint, either for $C^1$ flows on 3-manifolds, or for $C^1$ conservative flows on 3-manifolds. This means that we present some properties satisfied by a generic subset of all vector fields in compact 3-manifolds in the $C^1$ topology. We recall that a generic subset of a topological space is a countable intersection of open and dense subsets. Since $C^1(M)$, endowed with the $C^1$ topology, is a Baire space, then every generic subset in dense. The importance of generic properties stems from the fact that the intersection of any countable number of generic subsets is itself a generic subset, so that we can always add the properties we know to be generic. In this way we obtain topologically large families of vector fields with fairly strong dynamical properties.

The choice of the $C^1$ topology is a consequence of the present development of deep perturbation tools, such as the Closing and Connecting Lemmas, which are only available in the $C^1$ topology. However, dealing with the $C^1$ topology has serious disadvantages. This topology does not have a strong physical meaning when dealing with solutions of differential equations, since perturbations of solutions often arise naturally in higher topologies ($C^r$ with $r > 1$) by dealing with higher order derivatives. Moreover, some results that can be proved using the $C^1$ topology, such as the dichotomy we present in what follows for $C^1$ generic conservative vector fields in 3-manifolds, are simply not true in higher topologies, since the "KAM Theory" (see e.g. [26, 248]) directly contradicts this dichotomy; see below.

We first extend a classification result from hyperbolic dynamics to singular-hyperbolic attracting sets. The Spectral Decomposition Theorem for hyperbolic systems plays a central role. It ensures that an attracting hyperbolic set having dense periodic orbits must be a finite disjoint union of homoclinic classes. Here we provide a version of this result in the setting of singular-hyperbolic systems, presented in Sect. 5, following [170].

Theorem 9.1 An attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite union of transitive sets. Moreover, either the union is...
disjoint or the set contains finitely many distinct homoclinic classes. For $C^r$-generic flows, $r \geq 1$, the union is in fact disjoint.

We also show that a generic $C^1$ vector field on a closed 3-manifold either has infinitely many sinks or sources, or else is singular Axiom A without cycles. These results are contained in [168].

**Theorem 9.2** A generic vector field $X \in \mathcal{X}^1(M)$ satisfies (only) one of the following properties:
1. $X$ has infinitely many sinks or sources.
2. $X$ is singular Axiom A without cycles.

A Singular Axiom A vector field is such that the non-wandering set of the vector field has a decomposition into finitely many compact invariant sets $\Omega(X) = \Omega_1 \cup \cdots \cup \Omega_k$, each one being either a (uniformly) hyperbolic basic set (i.e. transitive, isolated and with a dense subset of periodic orbits) or a singular-hyperbolic attractor, or a singular-hyperbolic repeller with dense subset of periodic orbits (these are defined in Chap. 5—we note that, in this decomposition, the singular-hyperbolic sets are transitive by definition).

An analogous result was proved by Mañé in [145] for $C^1$ generic diffeomorphisms on surfaces.

Conservative flows are a traditional object of study from Classical Mechanics, see e.g. [26]. These flows preserve a volume form on the ambient manifold and thus come equipped with a natural invariant measure. On compact manifolds this provides an invariant probability giving positive measure (volume) to all open subsets. Therefore for vector fields in this class we have $\Omega(X) = M$ by the Recurrence Theorem. In particular such flows cannot have Lyapunov stable sets, either for $X$ or for $-X$.

The device of Poincaré sections has been used extensively in the previous chapters to reduce several problems, arising naturally in the setting of flows, to lower dimensional questions about the behavior of a transformation. In the opposite direction, recent breakthroughs in the understanding of generic volume preserving diffeomorphisms on surfaces have non-trivial consequences for the dynamics of generic conservative flows on three-dimensional manifolds.

The Bochi-Mañé Theorem [51] asserts that, for a $C^1$ residual subset of area preserving diffeomorphisms, either the transformation is Anosov, or the Lyapunov exponents are zero Lebesgue almost everywhere. This was announced by Mañé in [146] but only a sketch of a proof was available [149]. The complete proof, presented by Bochi, admits extensions to higher dimensions, obtained by Bochi and Viana in [53], stating in particular that either the Lyapunov exponents of a $C^1$ generic conservative diffeomorphism are zero Lebesgue almost everywhere, or the system admits a dominated splitting for the tangent bundle dynamics. A survey of this theory can be found in [52].

Using an important result of Zuppa in [277] proving that $C^{\infty}$ conservative flows are $C^1$ dense among conservative flows (which was recently generalized by Arbieto-
Matheus in [22] proving the $C^1$ denseness of Hölder-$C^1$ conservative flows, together with the zero volume results above and several delicate perturbation techniques adapted from the work of Bochi [51], we follow [19, 39] in Sect. 9.3 to prove the Bochi-Mané dichotomy for $C^1$ conservative flows on 3-manifolds: $C^1$ generically either the flow is Anosov or else the Lyapunov exponents are zero Lebesgue almost everywhere.

The presence of singularities imposes some differences between the discrete and continuous systems. More precisely, let $X^r_\mu(M)$ be the space of $C^r$ vector fields defining flows which preserve the volume form $\mu$ on $M$, for any $r \geq 1$, and let $X^r_\mu(M)^*$ be the subset of $X^r_\mu(M)$ of $C^r$ flows with zero divergence but without singularities.

**Theorem 9.3** There exists a residual set $\mathcal{R} \subset X^1_\mu(M)^*$ such that, for $X \in \mathcal{R}$, either $X$ is Anosov or else for Lebesgue almost every $p \in M$ all the Lyapunov exponents of $X^p$ are zero.

Developing the ideas of the proof of this result, the following statement on density of dominated splitting, now admitting singularities, was also obtained in the same work by Bessa [41].

Recall the definition of Linear Poincaré Flow in Sect. 2.6. Given an invariant subset $\Lambda$ for $X \in X^1(M)$, an invariant splitting $N^1 \oplus N^2$ of the normal bundle $N_\Lambda$ for the Linear Poincaré Flow $P^t$ is said to be $n$-dominated if there exists an integer $n$ such that we have the domination relation

$$\frac{\|P^n \mid N^1(p)\|}{\|P^n \mid N^2(p)\|} \leq \frac{1}{2},$$

for every $p \in \Lambda$.

**Theorem 9.4** There exists a dense set $\mathcal{D} \subset X^1_\mu(M)$ such that, for $X \in \mathcal{D}$, there exist invariant subsets $D$ and $Z$ whose union has full measure, such that

- for $p \in Z$, the flow has only zero Lyapunov exponents;
- $D$ is a countable increasing union $\Lambda_n$ of compact invariant sets admitting an $n$-dominated splitting for the Linear Poincaré Flow.

We prove these two results in Sect. 9.3. Now from Theorem 8.1 already proved in Chap. 8, we observe that $C^1$ generically the subset $D$ in the second possibility of the statement of Theorem 9.4 has positive volume if, and only if, the flow is Anosov. Hence we can extend Theorem 9.3 to flows with singularities.

**Theorem 9.5** There exists a generic subset $\mathcal{R} \subset X^1_\mu(M)$ such that for $X \in \mathcal{R}$

- either $X$ is Anosov;
- or else for Lebesgue almost every $p \in M$ all the Lyapunov exponents of $X^p$ are zero.
We remark the Kolmogorov-Arnold-Moser Theorem ensures the persistence of invariant circles with irrational rotations near an elliptic fixed point (the eigenvalues of the derivative of the map at the point are both complex with norm one) of a conservative diffeomorphism of a surface. In many cases the family of invariant circles has positive Lebesgue area for all $C^\infty$ nearby maps. Suspending this diffeomorphism we obtain a flow on a compact 3-manifold, which can be made incompressible, with a positive Lebesgue volume set consisting of invariant tori. The flow restricted to these bidimensional tori behaves like a linear irrational flow, without singularities, and with all Lyapunov exponents zero. Hence we obtain a positive Lebesgue measure invariant subset for a conservative flow on a 3-manifold, whose points have only zero Lyapunov exponents, and this is a persistent feature, valid for all nearby flows in the $C^r$ topology, for $r \geq 4$.

This clearly contradicts the statement of Theorem 9.5. So this $C^1$ generic result cannot be extended to higher topologies.

9.1 Spectral Decomposition

Here we prove Theorem 9.1, stating that an attracting singular-hyperbolic set with dense periodic orbits, and a unique singularity, is a finite union of transitive sets.

The straightforward extension of the result on a finite disjoint union of homoclinic classes from uniformly hyperbolic to a singular-hyperbolic attracting set with a dense subset of periodic orbits is false, as the next counterexample shows.

Consider a modification of the construction of the geometric Lorenz attractor given in Sect. 3.3, obtained by adding two equilibria to the flow located at $W^u(\sigma)$ as indicated in Fig. 9.1. This modification can be done in such a way that the new flow restricted to the cross-section $S$ has a $C^\infty$ invariant stable foliation and the quotient map in the leaf space is piecewise expanding with a single discontinuity $c$ as in the Lorenz case. The resulting attracting set can be proved to be a homoclinic class just as in the geometrical Lorenz case (see Sect. 3.3.6). In particular, such a set is transitive with dense periodic orbits and is also singular-hyperbolic by construction.

Fig. 9.1 A modified geometric Lorenz attractor
9.1 Spectral Decomposition

Fig. 9.2 A sketch of the construction of a singular-hyperbolic attractor which is not the disjoint union of homoclinic classes

Now glue two copies of this flow along the unstable manifold of the singularity $\sigma$ obtaining the flow depicted in Fig. 9.2. The resulting flow can be made $C^\infty$ easily.

In this way we construct an attracting singular-hyperbolic set with dense periodic orbits and three equilibria which is not the disjoint union of homoclinic classes (although it is the union of two transitive sets). It is possible to construct a similar counter-example with a unique singularity, while this counterexample has three equilibria. However, the construction in this case is more involved; see [38].

The above counterexample shows that, when dealing with the spectral decomposition for singular-hyperbolic sets, it is possible to obtain a finite union of transitive sets rather than a finite disjoint union of homoclinic classes. The next result shows that the former situation always occurs if the attracting set has only one singularity.

**Theorem 9.6** An attracting singular-hyperbolic set with dense periodic orbits and a unique singularity is a finite union of transitive sets.

**Proof** Split $\Lambda$ into finitely many connected components. On the one hand such components are clearly attracting with dense periodic orbits and the non-singular ones are hyperbolic, hence transitive, by the Spectral Theorem for uniformly hyperbolic sets; see e.g. [252]. On the other hand, the singular component satisfies the conditions of Theorem 6.38. Hence this component is either transitive or the union of two homoclinic classes, which are transitive sets. Therefore $\Lambda$, which is the union of its connected components, is a finite union of transitive sets.

Note that by a result of Morales [163] every transitive set of a flow $Y$, close to $X$, contained in the isolating neighborhood $U$ of a singular-hyperbolic attractor $\Lambda$ of $X$ must contain a singularity. Therefore, since compact invariant subsets in $\Lambda$ not containing singularities are hyperbolic and admit a spectral decomposition, and the number of singularities in $U$ is finite, the $\omega$-limit set in $U$ for $Y$ has finitely many transitive pieces only, all of which are singular. Hence, near a singular-hyperbolic attractor, the number of transitive pieces is robustly finite.
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It is natural to ask whether the union in Theorem 9.6 is disjoint. Recall that a vector field is Kupka-Smale if all its closed orbits are hyperbolic and their associated invariant manifolds are in general position; see Sect. 2.5.10.

**Theorem 9.7** An attracting singular-hyperbolic set, with dense periodic orbits and a unique singularity, of a Kupka-Smale vector field is a finite disjoint union of transitive sets.

**Proof** Let \( X \) be a Kupka-Smale vector field in a compact 3-manifold and let \( \Lambda \) be an attracting singular-hyperbolic set of \( X \) with dense periodic orbits and a unique singularity \( \sigma \). It suffices to prove that the connected component of \( \Lambda \) containing the singularity \( \sigma \) is transitive. By contradiction, suppose that this is not so.

On the one hand, by Theorem 6.44, we obtain a regular point \( a \) in the unstable manifold \( W^u(\sigma) \) of \( \sigma \) such that \( \omega(a) \) is a periodic orbit \( \mathcal{O}(p) \). On the other hand, the unstable manifold \( W^u(\sigma) \) is one-dimensional, and so the vector field exhibits a non-transverse intersection between \( W^u(\sigma) \) and \( W^s(\eta) \), contradicting the choice of \( X \) in the Kupka-Smale class. \( \square \)

Theorem 9.7 implies that the union in Theorem 9.6 is disjoint for most vector fields on closed 3-manifolds. Denote by \( \mathcal{R}'(M) \) the subset of all vector fields \( X \in \mathcal{X}'(M) \) for which every attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of \( X \) is a finite disjoint union of transitive sets. Standard \( C^1 \) generic arguments (see e.g. [56]) imply that \( \mathcal{R}'(M) \) is residual in \( \mathcal{X}'(M) \) when \( r = 1 \). The following corollary proves this assertion for all \( r \geq 1 \). The proof combines Theorem 9.7 with the classical Kupka-Smale Theorem (see e.g. [190]).

**Corollary 9.8** The class \( \mathcal{R}'(M) \) is residual in \( \mathcal{X}'(M) \) for every \( r \geq 1 \).

Now consider the complement of \( \mathcal{R}'(M) \). For a compact invariant subset \( \Lambda \) of a vector field \( X \) define the family \( \mathcal{G}(\Lambda) \) of homoclinic classes contained in \( \Lambda \). Note that if \( \Lambda \) is hyperbolic then \( \mathcal{G}(\Lambda) \) is finite. We are able to give sufficient conditions for the finiteness of \( \mathcal{G}(\Lambda) \) when \( \Lambda \) is a singular-hyperbolic set.

**Theorem 9.9** Let \( \Lambda \) be an attracting singular-hyperbolic set with dense periodic orbits and a unique singularity of \( X \in \mathcal{X}'(M) \). If \( \Lambda \) is not a disjoint union of transitive sets, then \( \mathcal{G}(\Lambda) \) contains finitely many homoclinic classes only.

Theorem 9.6 applies to the class of singular-hyperbolic vector fields introduced by Bautista in [36]. A vector field \( X \) is singular-hyperbolic if its non-wandering set \( \Omega(X) \) has dense critical elements and, if \( A(X) \) denotes the union of the attracting and repelling closed orbits, there is a disjoint union

\[
\Omega(X) \setminus A(X) = \Omega_1(X) \cup \Omega_2(X),
\]

where \( \Omega_1(X) \) is a singular-hyperbolic set for \( X \) and \( \Omega_2(X) \) is a singular-hyperbolic set for \(-X\).
This class of singular-hyperbolic vector fields contains the Axiom A vector fields (uniformly hyperbolic) and the singular Axiom A example resembling the geometric Lorenz attractor, described after Corollary 9.14.

Remark 9.10 An example of a singular-hyperbolic vector field in the 3-sphere $S^3$ which is not Kupka-Smale can be derived from the example described in Fig. 9.1: we consider the singularity $\sigma_1$, which is accumulated by regular orbits only on one side, and weaken its weak-contracting eigenvalue so that $DX(\sigma_1)$ now has three real eigenvalues $-\lambda, 0, \kappa$ with $\lambda, \kappa > 0$. We can perform this perturbation keeping the local stable manifold of $\sigma_1$ so that the global picture in Fig. 9.1 is kept. Since $\sigma_1$ becomes non-hyperbolic, the vector field is not Kupka-Smale.

The following is a direct consequence of Theorems 9.6 and 9.7, and taken together this completes the proof of Theorem 9.1.

Corollary 9.11 Let $X$ be a singular-hyperbolic vector field with a unique singularity on a compact 3-manifold. If $\Omega_1(X)$ is attracting and $\Omega_2(X)$ is repelling, then $\Omega(X)$ is a finite union of transitive sets. If $X$ is Kupka-Smale, then such a union is disjoint. In particular, the union is disjoint for a residual subset of vector fields in $X^c(M), r \geq 1$.

An example of a singular-hyperbolic vector field in $S^3$ satisfying the conditions of Corollary 9.11, without sinks nor sources, was described just before the statement of Corollary 9.12.

The extension of these results to general singular-hyperbolic attracting sets, with several singularities, is still work in progress.

Proof of Theorem 9.9 Suppose that $A$ is not a disjoint union of transitive sets. Split $A$ into finitely many connected components as before. It suffices to prove that $\mathcal{G}(A')$ contains finitely many homoclinic classes for all connected components $A'$ of $A$. On the one hand, for non-singular $A'$ we have nothing to prove, since $A'$ is uniformly hyperbolic by Proposition 6.2.

On the other hand, the singular connected component $A_0$ must contain $W^u(\sigma)$ (since it is connected), and $W^u(\sigma)$ has two connected components. Choose points $a, a'$ in each one. Observe that $A_0$ must not be transitive by the assumptions on $A$. Then by Theorem 6.44 there are periodic orbits such that $\omega(a) = \mathcal{O}(a)$ and $\omega(a') = \mathcal{O}(a')$. By contradiction assume that there are infinitely many distinct homoclinic classes in $A_0$.

Then there exists an infinite sequence of pairwise distinct periodic orbits $O_n \subset A_0$ and an infinite sequence $z_n \in O_n$, and so the set $A = \bigcup_n H(z_n)$ must contain $\sigma$. For otherwise $A \subset A_0 \setminus \{\sigma\}$ is uniformly hyperbolic and the number of homoclinic classes would be finite.

Consider then $x_n \in \mathcal{O}_n$ such that $x_n \xrightarrow{n \to +\infty} \sigma$. Since $x_n$ is not $\sigma$, the accumulation on $\sigma$ and the flow-boxes near $\sigma$ show that the orbit $\mathcal{O}_n$ accumulates also either $a$ or $a'$. Without loss of generality, assume the former case is true.
Fig. 9.3 The accumulation on one of the components of $W^s(a)$

Since $\omega_X(a) = \mathcal{O}(p)$ and $\mathcal{O}$ accumulates at $a$, we can find $z'_n \in \mathcal{O}$ passing close to $O$ as indicated in Fig. 9.3.

By the inclination lemma we can assume that $z'_n$ converges to a point either in one component $W^{u,+}$ of $W^s(O) \setminus \mathcal{O}$, or in the other component $W^{s,-}$. Again suppose that we are in the former case. By Lemma 6.43 and the inclination lemma we obtain $z'_n \in W^{u,+} = H^+$. But then $H(z'_n) = H(x) = H(z_n) = H^+$ for infinitely many $n$ (since Theorem 2.17 ensures that every homoclinic class contains a dense subset of periodic orbits, all of which are homoclinically related).

This contradicts the choice of $z_n$.

9.2 A Dichotomy for $C^1$ Generic 3-Flows

It is known that a generic non-singular vector field $X \in \mathcal{X}^1(M)$ either has infinitely many sinks or sources, or else is Axiom A without cycles; see Mañé [145] or Liao [133]. The robustness of the geometric Lorenz attractor obtained in Sect. 3.3 shows that this is not true in general if singularities are allowed. Allowing singularities we can improve this as follows. Let $\mathcal{W}^1(M) \subset \mathcal{X}^1(M)$ be the set of vector fields that cannot be $C^1$ approximated by homoclinic loops. The Connecting Lemma 2.20 implies that any singularity of every $X \in \mathcal{W}^1(M)$ is separated from the non-wandering set. Using the arguments of Wea [270] and Hayashi [108] we conclude that a generic vector field in $\mathcal{W}^1(M)$ either has infinitely many sinks or sources or else it is Axiom A without cycles.

Recently Arroyo and Hertz [28] proved that every vector field in $\mathcal{W}^1(M)$ either can be approximated by one that is Axiom A without cycles, or exhibits a homoclinic tangency associated to a periodic orbit.

9.2.1 Some Consequences of the Generic Dichotomy

Let us describe some consequences of Theorem 9.2. The first one is related to the abundance of three-dimensional vector fields exhibiting either attractors or repellors. As noted by Mañé in [145], a generic $C^1$ diffeomorphism in the 2-sphere $S^2$ does
exhibit either sinks or sources. It is then natural to ask whether such a result is valid for $C^1$ vector fields in the 3-sphere $\mathbb{S}^3$ instead of $C^1$ diffeomorphisms in $\mathbb{S}^2$. The answer is negative as the following example shows.

Write $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider in $\mathbb{R}^3$ an unknotted two-torus $T^2$. Then the closure in $\mathbb{S}^3$ of each connected component of $\mathbb{S}^3 \setminus T^2$ is a solid two-torus. Consider a Lorenz attractor in one solid two-torus and a Lorenz repeller in the other. Since a fundamental domain for the Lorenz attractor (respectively repeller) is an unknotted solid two-torus, we can glue the two solid two-torus through the unknotted torus, obtaining a flow in $\mathbb{S}^3$ whose non-wandering set equals the disjoint union of one Lorenz attractor and one Lorenz repeller. Such a flow is singular Axiom A, and it can not be approximated by vector fields with either sinks or sources. However from Theorem 9.2 we deduce

**Corollary 9.12** A generic vector field in $\mathcal{X}^1(M)$ does exhibit either attractors or repellers.

The second consequence of Theorem 9.2 is related to a conjecture by Palis in [189], see also Sect. 2.8, asserting the denseness of vector fields exhibiting a finite number of attractors whose basin of attraction forms a full Lebesgue measure subset. Theorem 9.2 gives an approach to this conjecture in the (open) set $\mathcal{G}^1(M)$ of $C^1$ vector fields on a closed 3-manifold $M$ which cannot be $C^1$ approximated by ones exhibiting infinitely many sinks or sources.

**Corollary 9.13** A generic vector field in $\mathcal{G}^1(M)$ exhibits a finite number of attractors whose basins of attraction form an open and dense subset of $M$.

This corollary follows from the no-cycle condition by the classical construction of filtrations adapted to the decomposition of the positive limit set of the flow, as the reader can easily see in [247, Chaps. 2 & 3].

Using the filtration to isolate the dynamics around each basic piece of the singular Axiom A decomposition, since the critical elements are robustly hyperbolic near to each basic piece (recall that singular-hyperbolicity is a robust property of the action of the flow on the tangent bundle), we obtain

**Corollary 9.14** A $C^r$ singular Axiom A flow without cycles is in $\mathcal{G}^r(M)$, the interior of the set of $C^r$ vector fields whose critical elements are hyperbolic, for any $r \geq 1$.

We note that there exists a classification by Hayashi [107] of the $C^1$ interior of the set of diffeomorphisms whose periodic points are hyperbolic: they are Axiom A without cycles.

The corresponding result for vector fields is false since the Lorenz attractor is not uniformly hyperbolic. Indeed, we note that we can easily construct a singular Axiom A vector field without cycles and with a singular basic set equivalent to the Lorenz attractor: just take the geometric Lorenz attractor constructed in Sect. 3.3, and embed and extend this flow to $\mathbb{S}^3$ with a repelling singularity at the "north pole" and a sink at the "south pole".
Proof of Theorem 9.2 The argument is based on the following result whose proof we postpone to Sect. 9.2. Denote by \( \mathcal{F}^r(M) \) the interior of the set of vector fields \( X \in \mathcal{X}^r(M) \) such that every periodic orbit and singularity of \( X \) is hyperbolic, for any \( r \geq 1 \).

**Theorem 9.15** Generic vector fields in \( \mathcal{F}^1(M) \) are singular Axiom A without cycles.

Following the arguments of Mañé in [145], we can obtain Theorem 9.2 from Theorem 9.15. Indeed, let \( \mathcal{S}^1(M) \subseteq \mathcal{X}^1(M) \) be the subset of \( C^1 \) vector fields such that every singularity of \( X \) is hyperbolic. Then \( \mathcal{S}^1(M) \) is open and dense in \( \mathcal{X}^1(M) \) by the local stability of hyperbolic critical elements. For \( X \in \mathcal{S}^1(M) \) define \( A(X) \) to be the set of periodic orbits and singularities of \( X \) that are sinks or sources.

The set-valued function \( \mathcal{X}^1(M) \ni X \mapsto A(X) \in \mathcal{P}(M) \) is lower semicontinuous, again by the local stability of hyperbolic critical elements, where \( \mathcal{P}(M) \) denotes the family of compact subsets of \( M \) endowed with the Hausdorff distance. Well known topological properties imply that the continuity points \( \mathcal{O} \) of this map form a residual subset of \( \mathcal{S}^1(M) \).

This ensures that every \( X \in \mathcal{O} \) not satisfying the first item of Theorem 9.2 is in \( \mathcal{F}^1(M) \).

Indeed, for \( X_0 \in \mathcal{O} \) with finitely many sinks and sources the set \( A(X_0) \) is a finite collection of critical elements of \( X_0 \). Assume by contradiction that \( X_0 \not\in \mathcal{F}^1(M) \).

Then we can find a \( C^1 \)-near vector field \( Y \) with a non-hyperbolic critical element \( \xi \). Hence \( \xi \) is away from a neighborhood of \( A(X_0) \). However \( \mathcal{O} \subseteq \mathcal{S}^1(M) \) and \( \mathcal{S}^1(M) \) is open, and thus we can take \( Y \in \mathcal{S}^1(M) \). This guarantees that \( \xi \) is not a singularity of \( Y \). Then the return map to a Poincaré section of the periodic orbit \( \xi \) has two eigenvalues, one of which has modulus 1. Perturbing \( Y \) we can find \( Z \in \mathcal{S}^1(M) \) arbitrarily \( C^1 \)-close to \( Y \) (and to \( X_0 \)) having either an attracting or repelling periodic orbit close to \( \xi \). This contradicts the continuity of the set map \( A(X) \) at \( X_0 \).

Now from Theorem 9.15 there exists a residual set \( \mathcal{R} \subseteq \mathcal{F}^1(M) \) such that every vector field in \( \mathcal{R} \) is singular Axiom A without cycles. The class

\[
\mathcal{D} = (\mathcal{O} \setminus \mathcal{F}^1(M)) \cup (\mathcal{O} \cap \mathcal{R})
\]

is residual in \( \mathcal{X}^1(M) \) by construction (recall that \( \mathcal{S}^1(M) \) is open and dense in \( \mathcal{X}^1(M) \)). Note that if \( X_0 \in \mathcal{D} \) does not satisfy the first item of Theorem 9.2, then \( X_0 \in \mathcal{O} \cap \mathcal{R} \), since \( X_0 \) cannot belong to \( \mathcal{O} \setminus \mathcal{F}^1(M) \) be the previous claim. This means that \( X_0 \) satisfies the second item of the statement of Theorem 9.2. \( \square \)

### 9.2.2 Generic 3-Flows, Lyapunov Stability and Singular-Hyperbolicity

Here we present a proof of Theorem 9.15. We use the auxiliary Theorems 9.16 and 9.17 below. Recall the definition and properties of Lyapunov stable sets in Sect. 2.5.10.
The first theorem ensures that transitive Lyapunov stable sets containing singularities, if not equal to a critical element, are $C^1$ generically singular-hyperbolic sets.

**Theorem 9.16** For generic vector fields $X \in \mathfrak{X}^1(M)$, every nontrivial transitive Lyapunov stable set with singularities of $X$ is singular-hyperbolic.

The second result provides a way to obtain a singular-hyperbolic attractor from a singularity belonging to a Lyapunov stable set of a generic three-dimensional vector field. Together with the previous results, it asserts that the unstable manifold of a singularity accumulates on a singular-hyperbolic set containing the singularity.

**Theorem 9.17** Every Lyapunov stable singular-hyperbolic set with dense singular unstable branches of $X \in \mathfrak{X}^1(M)$ is an attractor of $X$.

Here we say that a singular-hyperbolic set $\Lambda$ has dense singular unstable branches if $\Lambda = \omega(x)$ for all $x \in W^u(\sigma) \setminus \{\sigma\}$ and for every singularity $\sigma \in \Lambda$.

Now we explain how Theorem 9.15 is a consequence of Theorems 9.16 and 9.17, but first we need some preliminary results. The first one gives a sufficient condition for a transitive Lyapunov stable set with singularities to have singular unstable branches.

**Lemma 9.18** For generic vector fields $X \in \mathfrak{X}^1(M)$, a transitive Lyapunov stable set with singularities $\Lambda$ of $X$, such that the unstable manifold of every singularity in $\Lambda$ is one-dimensional, has dense singular unstable branches.

**Proof** Generically we can assume that $X \in \mathfrak{X}^1(M)$ satisfies the properties presented in Sect. 2.5.10 (in particular $X$ is Kupka-Smale). Let $\Lambda$ be a transitive Lyapunov stable set of $X$, $\sigma$ a singularity of $\Lambda$ and $q \in W^u(\sigma) \setminus \{\sigma\}$.

On the one hand, since $\Lambda$ is Lyapunov stable we have $W^u(\sigma) \subset \Lambda$ and in particular $\omega(q) \subset \Lambda$. On the other hand, we have $\dim(W^u(\sigma)) = 1$ by assumption. Then $\omega(q)$ is Lyapunov stable by Property L5 in Sect. 2.5.10.

But $\Lambda$ is transitive by construction and intersects $\omega(q)$, and so by Lemma 2.25 we have $\Lambda \subset \omega(q)$. Then $\omega(q) = \Lambda$ and $\Lambda$ has dense singular unstable branches as desired. □

The next one shows that the closure of the unstable manifold of a singularity accumulated by periodic orbits is transitive, provided that the unstable manifold is one-dimensional and its closure is Lyapunov stable.

**Lemma 9.19** Let $X \in \mathfrak{X}^1(M)$ and $\sigma \in \mathcal{S}(X) \cap \text{Per}(X)$ be such that $W^u(\sigma)$ is one-dimensional and $\omega(q)$ is Lyapunov stable for every $q$ in any of the branches of $W^u_X(\sigma) \setminus \{\sigma\}$. Then $\overline{W^u(\sigma)}$ is transitive.

**Proof** We have $W^u_X(\sigma) \setminus \{\sigma\} = \mathcal{O}(q_1) \cup \mathcal{O}(q_2)$ for every $q_1, q_2$ belonging to different connected components of $W^u(\sigma) \setminus \{\sigma\}$. 
On the one hand, since \( \sigma \in \overline{\text{Per}(X)} \) we can assume that \( q_1 \in \overline{\text{Per}(X)} \) without loss of generality. Then \( \omega(q_1) \subseteq \overline{\text{Per}(X)} \) by invariance. On the other hand, \( \omega(q_1) \) is Lyapunov stable for \( X \) by assumption. These two properties imply that \( \sigma \in \omega(q_1) \), since for \( p_n \in \text{Per}(X) \) with \( p_n \xrightarrow{n \to +\infty} q_1 \) we also have \( X^n(p_n) \to \sigma \) for some sequence \( t_n \to 0 \), and we can apply Lemma 2.25.

Therefore \( W^u(\sigma) \subseteq \omega(q_1) \) by the Lyapunov stability of \( \omega(q_1) \) once more. But \( W^u(\sigma) \supseteq \omega(q_1) \) by construction, and so we conclude that \( W^u(\sigma) = \omega(q_1) \). This shows that \( \overline{W^u(\sigma)} \) is transitive. \( \square \)

Using this we now show that any hyperbolic singularity accumulated by regular orbits of \( X \) is in a singular-hyperbolic attractor or repeller of the flow induced by \( X \).

**Theorem 9.20** For generic \( X \in \mathcal{X}^1(M) \) every \( \sigma \in S(X) \cap \overline{\text{Per}(X)} \) belongs to either a singular-hyperbolic attractor or a singular hyperbolic repeller.

**Proof** Let \( X \in \mathcal{X}^1(M) \) and let \( \sigma \) be as in the statement. Since \( X \) is generic we can assume that \( \sigma \) is hyperbolic. Note that \( \sigma \) must be of saddle-type, for otherwise \( \sigma \) is either a sink or a source, and in each case no periodic orbit would approach \( \sigma \). Hence either \( W^u(\sigma) \) or \( W^s(\sigma) \) is one-dimensional.

Suppose the former case is true. The latter case is the same for \( -X \). Define \( \Lambda = \overline{W^u(\sigma)} \). Property L3 in Sect. 2.5.10 implies that \( \Lambda \) is Lyapunov stable for \( X \) because \( X \) is generic. Property L5 then guarantees that we are in the setting of Lemma 9.19 and so \( \Lambda \) is transitive.

Therefore \( \Lambda \) is a nontrivial transitive Lyapunov stable set of \( X \). As \( X \) is generic, Theorem 9.16 ensures that \( \Lambda \) is singular-hyperbolic. By Theorem 5.10 we know that every singularity in \( \Lambda \) has one-dimensional unstable manifold. We conclude that \( \Lambda \) has dense singular unstable branches by Lemma 9.18, since \( X \) is generic. Then \( \Lambda \) is an attractor by Theorem 9.17. \( \square \)

Now we have the tools to complete the proof of Theorem 9.15 using all the previous results which assume Theorems 9.16 and 9.17.

**Proof of Theorem 9.15** For \( X \in \mathcal{X}^1(M) \) denote by \( S^*(X) = S(X) \cap \overline{\text{Per}(X)} \) the (finite) set \( \{ \sigma_1, \ldots, \sigma_k \} \) of singularities accumulated by periodic orbits of \( X \).

Theorem 9.20 ensures that for generic \( X \in \mathcal{X}^1(M) \) and for every \( i = 1, \ldots, k \) there is a compact invariant set \( \Lambda_i \) of \( X \) such that \( \sigma_i \in \Lambda_1 \), and \( \Lambda_i \) is either a singular-hyperbolic attractor or a singular-hyperbolic repeller of \( X \).

We claim that \( H^* = \Omega(X) \setminus \bigcup_{i=1}^k \Lambda_i \) is a finite disjoint union of uniformly hyperbolic basic sets. Indeed \( H^* \setminus S(X) \) is closed in \( M \), for otherwise we can find a sequence of regular points \( x_n \) in \( H^* \) converging to some singularity \( \sigma \in S(X) \setminus S^*(X) \). But Property L2 gives \( \Omega(X) = \overline{\text{Per}(X)} \cup S(X) \), and so \( \sigma \) is accumulated by periodic orbits because \( S(X) \) is finite. Hence \( H^* \setminus S(X) \) is a closed invariant subset of \( X \) without singularities. It is known, after Wen [270], that \( C^1 \)
generically such sets are uniformly hyperbolic. Property L.2 again ensures that $H^s = \text{Per}(X) \cap H^s \cup S(X) \setminus S^u(X)$. The Spectral Decomposition Theorem for uniformly hyperbolic sets now guarantees that $H^s$ decomposes into finitely many basic pieces, together with finitely many singularities.

From this we see that $\mathcal{O}(X)$ splits into a disjoint union of compact invariant sets each one being either a hyperbolic basic set or a singular-hyperbolic attractor, or a singular-hyperbolic repeller. Hence $X$ is a singular Axiom A vector field. For generic $X$ we can also assume that the vector field is Kupka-Smale, and thus there are no cycles between the transitive pieces in the above decomposition. The proof of Theorem 9.15 is complete depending on Theorems 9.16 and 9.17.

Proof of Theorem 9.16 Recall that there exists a residual subset $\mathcal{O}$ of the family $C^1(M)$ of vector fields whose singularities are hyperbolic, such that the map $X \in C^1(M) \mapsto \Lambda(X)$ restricted to $\mathcal{O}$ is continuous (see the arguments after the statement of Theorem 9.15). Define $\mathcal{R} = \mathcal{O} \cap S^1(M)$ which is residual in $S^1(M)$.

Given $X \in \mathcal{R}$ and $\sigma \in S(X) \cap \Lambda$ for a non-trivial attractor $\Lambda$, observe that every vector field $Y$ sufficiently $C^1$-close to $X$ has no sources or sinks near to $\Lambda$, for otherwise we deduce a contradiction to the choice of $X$ in the continuity set $\mathcal{O}$. All the critical elements of $Y$ are also hyperbolic. Then $Y$ is in the setting of Theorem 2.33, and thus the Linear Poincaré Flow over $\Lambda \setminus S(X)$ is robustly dominated. This means that $\Lambda$ is in the setting of Lemmas 5.22 and 5.30. Thus we deduce that, for $X \in \mathcal{R}$, if $\sigma \in S(X)$ belongs to a non-trivial attractor $\Lambda$ of $X$, then $\sigma$ is Lorenz-like for $X$ and $\mathcal{W}^u(\sigma) \cap \Lambda = \{\sigma\}$.

Now let $X \in \mathcal{R}$ have a non-trivial transitive Lyapunov stable set $\Lambda$ containing a singularity. The previous arguments ensure that $\Lambda$ is in the setting of Theorem 5.34, and hence $\Lambda$ is a singular-hyperbolic attractor.

Proof of Theorem 9.17 We need the following sufficient condition for a Lyapunov stable singular-hyperbolic set, with dense singular unstable branches, to be an attractor.

Lemma 9.21 Let $\Lambda$ be a Lyapunov stable singular-hyperbolic set with dense singular unstable branches of $X \in \mathcal{C}^r(M)$, $r \geq 1$. If $\Lambda$ admits an adapted cross-section $\Sigma$ such that every point in the interior of $\Sigma$ belongs to the stable leaf of some point of $\Lambda \cap \Sigma$, then $\Lambda$ is an attractor.

Proof From Lemma 2.26 it is enough to prove that, if $x_n$ is a sequence converging to some point $p \in \Lambda$, then $\omega(x_n)$ is contained in $\Lambda$ for every big enough $n$. Now $\omega(p)$ satisfies one of the following alternatives.

1. $\omega(p)$ contains a singularity $\sigma$ of $\Lambda$.

The orbits of $x_n$ will have $\sigma$ as an accumulation point. Hence the orbit of $x_n$ also accumulates at some regular point of the unstable manifold of $\sigma$. Since $\omega(q) = \Lambda$ by assumption, we see that for every big enough $n$ the orbit of $x_n$
crosses the interior of \( \Sigma \). Then by the assumption on \( \Sigma \) we get \( y \in \Lambda \) such that \( \omega(x_n) \subset W^s(y) \), that is, \( \omega(x_n) \subset \Lambda \) for all sufficiently big \( n \).

2. \( \omega(p) \) is far from singularities.

Take \( S \) an adapted cross-section to a point \( q \) of \( \omega(p) \). Then for all big enough \( n \) the orbit of \( x_n \) crosses the interior of \( S \) at some point \( x'_n \) very close to \( q \). Since \( \omega(p) \) is uniformly hyperbolic by Proposition 6.2, the unstable manifold of \( q \) is well defined and \( \mathcal{W}^u(q) \cap S \) is a line in \( S \) crossing all stable manifolds of \( S \) in a neighborhood of \( q \). Then \( x'_n \) belongs to some of these stable lines. Since \( \mathcal{W}^u(q) \) is inside \( \Lambda \) by Lyapunov stability, we see that \( x_n \) belongs to the stable manifold of some point of \( \Lambda \). Again \( \omega(x_n) \subset \Lambda \) for all sufficiently big \( n \).

Now suppose that \( \Lambda \) is not an attractor. Then by Lemma 9.21, given any regular point \( x \in \Lambda \), we can find an adapted cross-section \( \Sigma' \) such that the intersection \( \Lambda \cap \Sigma' \) is contained in the interior of \( \Sigma \). Indeed, \( \Sigma \cap \Lambda \) contains \( z_0 \) such that \( W^s(z_0, \Sigma) \) does not touch \( \Lambda \), and then one of the connected components of \( \Sigma \setminus W^s(z_0, \Sigma) \), which is also an adapted cross-section containing \( x \), contains \( z_1 \) such that \( W^s(z_1, \Sigma) \cap \Lambda = \emptyset \). The strip \( \Sigma' \) between \( W^s(z_0, \Sigma) \) and \( W^s(z_1, \Sigma) \) only intersects \( \Lambda \) in its interior.

Cover \( \Lambda \) by finitely many flow-boxes near singularities and tubular flow-boxes through adapted cross-sections, around regular pieces of \( \Lambda \), just as in Chap. 7, but with the family \( \mathcal{S} \) of adapted cross-sections chosen so that \( \Lambda \cap \mathcal{S} \subset \text{int}(\mathcal{S}) \).

Observe that, since \( \Lambda \) is Lyapunov stable, we can find a neighborhood \( U \) of \( \Lambda \) such that \( U \cap \mathcal{S} \subset \text{int}(\mathcal{S}) \) and then another neighborhood \( V \subset U \subset \Lambda \) satisfying \( X'(V) \subset U \) for all \( t > 0 \). Then the Poincaré map \( R \) defined as in Sect. 6.1 between the sections of \( \mathcal{S} \) admits only finitely many discontinuity points, at the intersection of \( \mathcal{S} \) with a compact part of the stable manifolds of the singularities of \( \Lambda \), since its image cannot touch the boundary of \( \mathcal{S} \). We can choose the "waiting time" \( \tau \) of \( R \) so that the expansion rate on center-unstable cones is at least 4.

Let \( \mathcal{S}^* \) be the subset of ingoing cross-sections near singularities of \( \mathcal{S} \). Fix a point \( x_0 \in \Lambda \cap \mathcal{S}^* \setminus \bigcup \{ W^s(\sigma) : \sigma \in \mathcal{S} \setminus \Lambda \} \) and a connected \( cu \)-curve \( y_0 \) inside \( \mathcal{S}^* \) through \( x_0 \) not touching the lines of intersection of \( \mathcal{S}^* \) with the local stable manifold of the singularities. The image curve \( R^i(y_0) \), for \( i > 0 \), is well defined until it returns to \( \mathcal{S}^* \), because the image of \( R \) does not fall outside of \( \text{int}(\mathcal{S}) \). Let \( y_2 \) be the next return to \( \mathcal{S}^* \). Then its length \( \ell(y_2) \) is at least \( 4 \cdot \ell(y_1) \).

The image of \( y_2 \) is well defined except perhaps at \( y_2 \cap W^u_{loc}(\sigma) \) for some singularity \( \sigma \) of \( \Lambda \). In this case we replace \( y_2 \) by the longest connected component of \( y_2 \setminus W^u_{loc}(\sigma) \). Then \( \ell(y_2) \geq 2 \cdot \ell(y_1) \).

Inductively we obtain a sequence \( \gamma_n, n \geq 1 \), of larger and larger \( cu \)-curves in the interior \( \mathcal{S}^* \), which is a finite collection of bounded cross-sections. Since the \( cu \)-curves cannot be tangent to the stable foliation, and so cannot curl inside \( \mathcal{S} \), this is impossible.

This contradiction shows that \( \Lambda \) must be an attractor and concludes the proof of Theorem 9.17. \( \square \)
9.3 Lyapunov Exponents of $C^1$ Generic Incompressible 3-Flows

Here we prove Theorems 9.3 and 9.4. Since the proof is rather technical, we present first an outline of the strategy.

Recall the definition of dominated splitting for the Linear Poincaré Flow and that the splitting $N^1 \oplus N^2$ of the normal bundle $N$ is an $m$-dominated splitting for the Linear Poincaré Flow if it is $P^1_X$-invariant and there is an uniform $m \in \mathbb{N}$ such that

$$\Delta(p, m) = \frac{\| P^m_X(p) N^1_p \|}{\| P^m_X(p) N^2_p \|} \leq \frac{1}{2}, \quad (9.1)$$

for any point $p \in \Lambda$.

We define some useful $X'$-invariant sets:

- $\mathcal{O} = \{\text{Oseledec regular points}\}$, has full Lebesgue measure in $M$, since a volume form is assumed to be invariant (see Sect. 2.7.2);
- $\mathcal{O}^+ = \{p \in \mathcal{O}: \text{the orbit of } p \text{ has positive Lyapunov exponent}\}$;
- $\mathcal{O}^0 = \{p \in \mathcal{O}: \text{the orbit of } p \text{ has only zero Lyapunov exponents}\}$;
- $\Lambda_m(X) := \{p \in \mathcal{O}^+: p \text{ has an } m\text{-dominated splitting for the Linear Poincaré Flow}\}$;
- $\Gamma^m_n(X) := M \setminus \Lambda_m(X)$;
- $\Gamma^+_m(X) := \mathcal{O}^+(X) \setminus \Lambda_m(X)$;
- $\Gamma^+_m(X) := \{p \in \Gamma^m_n(X): p \notin \text{Per}(X)\}$.

The set of Oseledec regular points where (9.1) does not hold will be denoted by $\Delta_m(X)$. Clearly, in $\Delta_m(X)$, the orbits do not have an $m$-dominated splitting. Nevertheless for some $p \in \Delta_m(X)$ there might still exist some iterate $X'(p)$ where (9.1) holds. Taking this into account we note that $\Gamma^+_m(X) = \bigcup X'(\Delta_m(X))$.

Let $\lambda^+(X, p)$ be the upper Lyapunov exponent which exists for $\mu$-almost every $p \in M$ by the Theorem of Oseledec; see Sect. 2.7.1 and [147]. When there is no ambiguity we denote $\lambda^+(X, p)$ by $\lambda^+(p)$. Given an $X'$-invariant set $\Gamma \subset M$ we define the "entropy function" by

$$LE : \mathcal{X}^1_\mu(M) \to [0, +\infty), \quad X \mapsto \int_M \lambda^+(p) d\mu(p).$$

The next lemma gives an equivalent expression for this function.

Lemma 9.22 Let $\Gamma \subset M$ be an $X'$-invariant subset. Then

$$LE(X, \Gamma) = \int_{\Gamma} \lambda^+(p) d\mu(p) = \inf_{n \geq 1} \frac{1}{n} \int_{\Gamma} \log \| P^n_X(p) \| d\mu(p).$$

In particular, this shows that $LE(X) = LE(X,\ M)$ is upper semicontinuous as a function of $X$.

Proof Using the definition of $\lambda^+(p)$ we write $\int_{\Gamma} \frac{1}{n} \int \log \| P^n_X(p) \| d\mu(p)$ as

$$\lim_{n \to +\infty} \frac{1}{n} \int \log \| P^n_X(p) \| d\mu(p) = \lim_{n \to +\infty} \frac{1}{n} \int \log \| P^n_X(p) \| d\mu(p) = I_n(X)$$
and the sequence $I_n(X)$ is subadditive: $I_{p+q}(X) \leq I_p(X) + I_q(X)$ for all $p, q \in \mathbb{Z}^+$. Thus it satisfies $\lim_{n \to +\infty} (I_n(X)/n) = \inf_{n \geq 1} (I_n(X)/n)$.

Denote $LE(X, M)$ by $LE(X)$. The next proposition will be crucial to prove Theorem 9.3.

**Proposition 9.23** Let $X \in X^1_\mu(M)^*$, with $X^1$ aperiodic, and suppose that any hyperbolic set has zero Lebesgue measure. Let $\varepsilon, \delta > 0$ be given. Then there exists a $C^1$ zero divergence vector field $Y$ which is $\varepsilon$-$C^1$-close to $X$, such that $LE(Y) < \delta$.

We assume Proposition 9.23 and prove Theorem 9.3 first. By Corollary 8.2, we have a dense set such that every $X$ is $C^2$ aperiodic and with hyperbolic sets having full or zero measure. The set of conservative Anosov vector fields, denoted by $\mathcal{A}$, is open. For all $k \in \mathbb{N}$ the set $\mathcal{A}_k = \{ X \in X^1_\mu(M)^* : LE(X) < k^{-1} \}$ is open, by Lemma 9.22. $LE$ is upper semicontinuous. By Proposition 9.23, with $\delta = k^{-1}$, we get $\mathcal{A}_k$ dense in $\mathcal{A}^e$, and so the set $\mathcal{R} = \bigcap_k \mathcal{A} \cup \mathcal{A}_k$ is a $C^1$ residual set. But $\mathcal{R} = \mathcal{A} \cup \bigcap_k \mathcal{A}_k = \mathcal{A} \cup \{ X \in X^1_\mu(M)^* : LE(X) = 0 \}$, and therefore for $X \in \mathcal{R}$ either $X$ is an Anosov vector field or $LE(X) = \int_M \lambda^+ (p) d\mu(p) = 0$. This last equality implies that $\mu$-almost every $p \in M$ has zero Lyapunov exponents and Theorem 9.3 is proved.

To prove Proposition 9.23, we consider a large $m \in \mathbb{N}$ (depending on $\varepsilon$) and we use the fact that almost every orbit does not have an $m$-dominated splitting, for otherwise by Corollary 8.2 $X$ must be Anosov. We start with a local argument and take $p \in T^m_m(X)$, and also $t \gg m$. By a recurrence result (see Lemma 9.40), we obtain $q \approx X^{1/2}(p)$ such that $\Delta(q, m) \geq 1/2$, say $\varepsilon \in \Delta_m(X)$. In Sect. 9.3.3 we use the $\varepsilon$-$C^1$-perturbation $Y$ of $X$, developed previously in Sects. 9.3.1 and 9.3.2, to map the direction $N^s_q$ into $N^s_{X^e(q)}$. In Sect. 9.3.4 we conclude that this argument allows us to prove that for most points $q$ near $p$ the norm of $P^Y_q(q)$ is smaller than $\delta$. The formula for $LE$ in Lemma 9.22 allows us to compute a bound for $LE(Y)$ in a finite time $t$. Finally, in Sect. 9.3.5, extend this local procedure to the whole of $M$ through a Kakutani tower argument.

### 9.3.1 Conservative Tubular Flow Theorem

The following theorem, due to Dacorogna and Moser [75], will be used to obtain a conservative local change of coordinates which trivializes a vector field.

**Theorem 9.24** (Dacorogna-Moser) Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ with $C^5$ boundary $\partial \Omega$ and $g, f : \partial \Omega \to \mathbb{R}$ positive functions of class $C^s$ ($s \geq 2$). Then there exists a diffeomorphism $\phi : \Omega \to \phi(\Omega) \subseteq \mathbb{R}^n$ of class $C^s$ which satisfies the partial differential equation:

$$g(\phi(q)) \det(D\phi_q) = \lambda f(q),$$

for all $q \in \Omega$ where $\lambda = \int g d\mu / \int f d\mu$. We also have $\phi = \text{Id}$ at $\partial \Omega$. 

Let \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) be the constant vector field defined by \( T(x, y, z) = (c, 0, 0) \) for some \( c > 0 \) and let \( \mathcal{F}, \mathcal{C} \) be the flow-boxes \( \mathcal{F}_\mathcal{C}(p)(B(p, r)) \) and \( \mathcal{F}_\mathcal{C}(\bar{p})(B(\bar{p}, r)) \). We start by giving a brief and informal idea of the proof of Lemma 9.25 below. We would like to find a \( C^1 \) volume-preserving diffeomorphism \( \hat{\psi} : \mathcal{C} \to \mathcal{F} \) such that \( X = \hat{\psi} T \). Given a map \( \psi : B(\bar{p}, r) \to B(p, r) \) and \( w \in \mathcal{C} \), let \( t_w \) be such that \( T^{-t_w} \in B(\bar{p}, r) \). We define \( \psi(w) := X^{t_w}(\psi \circ T^{-t_w}(w)) \). It is clear that, if \( \psi \) is volume-preserving, then \( \hat{\psi} \) will be a composition of three volume-preserving maps and we are done. However, the choice of \( \psi \) to be volume-preserving needs some care and to achieve our purposes we use Theorem 9.24 in an appropriate way.

**Lemma 9.25** (Conservative flow-box theorem) Given a vector field \( X \in \mathcal{X}_{\mathbb{R}}(M) \) (for \( s \geq 2 \)) and a non-singular point \( p \in M \) (eventually periodic with period \( \tau > 1 \)), there exists a conservative \( C^1 \) diffeomorphism \( \Psi : \mathcal{F} \to \mathcal{C} \) such that \( T = \Psi p \).

**Proof** Assume that \( p = 0 \) and \( \mathcal{X}(0) \subseteq \{(x, 0, 0) : x \in \mathbb{R}\} \). Let \( X_1(x, y, z) \) be the projection into the first coordinate of \( X(x, y, z) \). For a small \( r > 0 \) we define the functions \( f : B(\bar{p}, r) \to \mathbb{R} \) and \( g : B(p, r) \to \mathbb{R} \) such that \( f(\bar{y}, \bar{z}) := 1 \) for \( (\bar{y}, \bar{z}) \in B(\bar{p}, r) \) and \( g(y, z) := X_1(0, y, z) \) for \( (y, z) \in B(p, r) \) (see Fig. 9.4). Since \( g \) is of class \( C^1 \), we apply Theorem 9.24 to \( \Omega = B(p, r) \subseteq \mathbb{R}^2 \) and obtain a diffeomorphism \( \varphi : \mathcal{O} \to \varphi(\mathcal{O}) \subseteq \mathbb{R}^2 \) of class \( C^1 \) satisfying the partial differential equation \( g(\varphi(y, z)) \det(D\varphi_{x,z}) = \lambda \), for all \( (y, z) \in \mathcal{O} \), where \( \lambda = \int g \, d\mu / \int 1 \, d\mu \) and \( \varphi|_{\partial \mathcal{O}} = Id \).

Now we define the \( C^1 \) change of coordinates by

\[
\hat{\psi} : \mathbb{R} \times \mathcal{O} \to \mathbb{R}^3, \quad (\bar{y}, \bar{z}) \mapsto X^{\lambda^x}(0, \varphi(\bar{y}, \bar{z})).
\]

First we claim that

\[
\det(D\hat{\psi}(0, \bar{y}, \bar{z})) = 1 \quad \text{for all } (0, \bar{y}, \bar{z}) \in \mathbb{R} \times \mathcal{O}.
\]  

(9.3)

By a straightforward computation of the derivative, we obtain

\[
\det(D\hat{\psi}(0, \bar{y}, \bar{z})) = \begin{vmatrix}
\lambda^{-1} X_1(0, y, z) & 0 & 0 \\
\lambda^{-1} X_2(0, y, z) & \frac{\partial \varphi_y}{\partial y}(\bar{y}, \bar{z}) & \frac{\partial \varphi_z}{\partial y}(\bar{y}, \bar{z}) \\
\lambda^{-1} X_3(0, y, z) & \frac{\partial \varphi_y}{\partial z}(\bar{y}, \bar{z}) & \frac{\partial \varphi_z}{\partial z}(\bar{y}, \bar{z}) 
\end{vmatrix}
\]

**Fig. 9.4** The conservative change of coordinates straightening out all orbits.
Now using (9.2) of Theorem 9.24 we conclude that
\[
\det \left( D\hat{\psi}(0, \bar{y}, \bar{z}) \right) = \lambda^{-1} X_1(0, y, z) \det \left( D\varphi(\bar{y}, \bar{z}) \right) = g(y, z) \lambda^{-1} \det \left( D\varphi(\bar{y}, \bar{z}) \right) = 1,
\]
and therefore (9.3) is proved. Let us now check that
\[
\det \left( D\hat{\psi}(\bar{x}_0, \bar{y}_0, \bar{z}_0) \right) = 1 \quad \text{for all } \left( \bar{x}_0, \bar{y}_0, \bar{z}_0 \right) \in \mathcal{C}.
\]
We note that
\[
\hat{\psi}(\bar{x}, \bar{y}, \bar{z}) = X^{\lambda^{-1} \bar{x}_0} \left[ X^{\lambda^{-1}(\bar{y} - \bar{x}_0)}(0, \varphi(\bar{y}, \bar{z})) \right] = X^{\lambda^{-1} \bar{x}_0} \left[ \hat{\psi}(\bar{x} - \bar{x}_0, \bar{y}, \bar{z}) \right],
\]
so that
\[
D\hat{\psi}(\bar{x}, \bar{y}, \bar{z}) = D X^{\lambda^{-1} \bar{x}_0} \hat{\psi}(\bar{x} - \bar{x}_0, \bar{y}, \bar{z}) \circ D\hat{\psi}(\bar{x} - \bar{x}_0, \bar{y}, \bar{z}).
\]
Evaluating \( D\hat{\psi}(\bar{x}, \bar{y}, \bar{z}) \) at \( \bar{x} = \bar{x}_0 \) we get
\[
D\hat{\psi}(\bar{x}_0, \bar{y}, \bar{z}) = D X^{\lambda^{-1} \bar{x}_0} \circ D\hat{\psi}(0, \bar{y}, \bar{z}).
\]
Now we use (9.3) and the fact that the flow \( X^t \) is volume-preserving to conclude that
\[
\det(D\hat{\psi}(\bar{x}_0, \bar{y}_0, \bar{z}_0)) = 1.
\]
Finally, we take \( c := \lambda \) and consider the constant vector field \( T := (\lambda, 0, 0) \). Let \((x, y, z) = \hat{\psi}(\bar{x}, \bar{y}, \bar{z}) \). By a simple computation, we deduce that
\[
\hat{\psi}_* T(x, y, z) = D\hat{\psi}(\bar{x}, \bar{y}, \bar{z})(T(\bar{x}, \bar{y}, \bar{z})) = \left( X_1 \left( X^{\lambda^{-1} \bar{x}_0}(0, y, z) \right), X_2 \left( X^{\lambda^{-1} \bar{x}_0}(0, y, z) \right), X_3 \left( X^{\lambda^{-1} \bar{x}_0}(0, y, z) \right) \right) = X \left( \hat{\psi}(\bar{x}, \bar{y}, \bar{z}) \right).
\]
Taking \( \psi = \hat{\psi}^{-1} \) we obtain \( T = \psi_* X \). \( \square \)

### 9.3.2 Realizable Linear Flows

The next definition adapts the definition of realizable sequence given by Bochi in [51] and will also be central in the proof of our theorem. In broad terms we consider modified area-preserving linear maps acting in the normal bundle at \( p \), \( L'(p) : N_p \to N_{X'(p)} \) that do exactly what we want. Finally, we ask whether these maps are \((\gamma\text{-almost } C^1)\) realizable as the Linear Poincaré Flow of \( Y \), \( \varepsilon\text{-}C^1\)-close to \( X \), computed on small transversal neighborhoods of one point.

We need to fix some notations before the statement. We recall that the Linear Poincaré Flow is the differential of the standard Poincaré map
\[
\mathcal{P}_X^p(p) : \mathcal{Y}_p \subset N_p \to \mathcal{N}_{X'(p)},
\]
where \( \mathcal{N}_{X'(p)} \), for \( s = 0, t \), is a surface contained in \( M \) whose tangent space at \( X^s(p) \) is the normal direction \( N_{X'(p)} \) for \( s = 0, t \) and \( \mathcal{Y}_p \) is a small neighborhood of \( p \). We can always guarantee the existence of a continuous time-\( t \) arrival function
\( \tau(\pi, t) \) from \( \mathcal{V}_p \) into \( \mathcal{N}_p \) by using the implicit function theorem. Due to the presence of singularities, \( \mathcal{V}_p \) may be very small.

Given the Poincaré map \( \mathcal{P}_X^\alpha(p) : \mathcal{V}_p \subseteq \mathcal{N}_p \to \mathcal{N}_X^\alpha(p) \), where \( \mathcal{V}_p \) is chosen sufficiently small, and given \( B \subseteq \mathcal{V}_p \), the set

\[
\mathcal{P}_X^\alpha(p)(B) := \{ \mathcal{P}_X^\alpha(p)(q) : q \in B, t \in [0, n] \} = \mathcal{P}_X^\alpha(p)_X(B)
\]

is called the time-\( n \) length flow-box at \( p \) associated to the vector field \( X \). We remark that the sections \( (\mathcal{P}_X^\alpha(p)(B))_{t \in [0, n]} \) are pairwise disjoint if \( \mathcal{V}_p \) is small enough.

We include here the following useful result that will be crucial later; this enables us to estimate the distortion of the area form pushed-forward between cross-sections by Poincaré maps. Given \( n_1, n_2 \in \mathcal{T}_q \mathcal{N}_p \) for \( q \in \mathcal{N}_p \), we can define a pair of 2-forms induced by the volume form \( \omega \) according to

\[
\hat{\omega}_q(n_1, n_2) := \omega_q(X(q), n_1, n_2) \quad \text{and} \quad \bar{\omega}_q(n_1, n_2) := \omega_q(X(q), \frac{n_1}{\|X(q)\|}, n_2).
\]

It turns out that \( (\mathcal{P}_X^\alpha(p))^* \hat{\omega}_p = \bar{\omega}_{X \circ \phi_t(q)} \) for all \( q \in \mathcal{N}_p \). The measure \( \bar{\mu} \) induced by the 2-form \( \bar{\omega} \) is not necessarily \( \mathcal{P}_X^\alpha \)-invariant; however, both measures \( \hat{\mu} \) and \( \bar{\mu} \) are equivalent. We call \( \bar{\mu} \) the Lebesgue measure at normal sections or modified area.

In fact, given \( n_1, n_2 \in \mathcal{N}_p \) we have

\[
(\mathcal{P}_X^\alpha(p))^* \bar{\omega}_p(n_1, n_2) = x(t)^{-1} \bar{\omega}_{X^t(p)}(n_1, n_2),
\]

where \( x(t) = \|X(X^t(p))\|/\|X(p)\|^{-1} \). Since the flow is volume-preserving we have \( |\det P^\alpha_X(p)| = x(t)^{-1} \). Therefore we can give an explicit expression for the infinitesimal factor of area distortion by the Linear Poincaré Flow, which in turn implies the following simple lemma.

**Lemma 9.26** Given \( \nu > 0 \) and \( T > 0 \), there exists \( r > 0 \) such that for any measurable set \( K \subseteq B(p, r) \subseteq \mathcal{N}_p \) we have \( |\bar{\mu}(K) - x(t) \cdot \bar{\mu}(\mathcal{P}_X^\alpha(p)(K))| < \nu \) for all \( t \in [0, T] \).

This provides a way to estimate the area distortion due to the Poincaré map between cross-sections on a very small neighborhood around a piece of orbit of the flow (for a proof see [39, Lemma 3.1.3]).

**Definition 9.27** Given \( X \in \mathcal{X}_\mu^\alpha(M), \nu > 0, 0 < \kappa < 1 \) and a non-periodic point \( p \), we say that the modified area-preserving sequence of linear maps \( L_j : \mathcal{N}_X^j(p) \to \mathcal{N}_X^{j+1}(p) \) for \( j = 0, \ldots, n-1 \) is an \((\nu, \kappa)-\text{realizable linear flow of length} \ n \) at \( p \) if, for all \( \gamma > 0 \), there exists \( r > 0 \) such that for any open set \( \emptyset \neq U \subseteq B(p, r) \subseteq \mathcal{N}_p \) we can find

(a) a measurable set \( K \subseteq U \) such that \( \bar{\mu}(K) > (1 - \kappa) \bar{\mu}(U) \), and

(b) a zero divergence vector field \( Y, \| \cdot \| \)-close to \( X \), such that:

(i) \( Y^t = X^t \) outside \( \mathcal{P}_X^\alpha(p)(U) \) and \( DX_q = DY_q \) for every \( q \in U, \mathcal{P}_X^\alpha(p)(U) \);

(ii) if \( q \in K \), then \( \|P^\alpha_Y(q) - L_j\| < \gamma \) for \( j = 0, 1, \ldots, n-1 \).
In the previous definition we consider integer iterates, but there is no restriction to consider any intermediate linear maps, such as \( L_j : N_{X^{t_j}(p)} \to N_{X^{t_{j+1}}(p)} \) with \( t_j < t_{j+1} \) and \( \sum_{j=0}^{n-1} t_j = n \). The point \( p \) may also be periodic, but with period larger than \( n \). The realizability we deal here is with respect to the \( C^1 \) topology.

Next we show how to build some elementary realizable linear flows: the Linear Poincaré Flow itself and also the juxtaposition of two realizable linear flows are realizable linear flows.

**Lemma 9.28** Let \( X \in \mathcal{X}_L^1(M) \) and \( p \in M \) a non-periodic point.

1. For each \( t \in \mathbb{R} \), \( \mathcal{P}_X^t(p) \) is \( (\varepsilon, \kappa) \)-realizable of length \( t \) for every \( \varepsilon \) and \( \kappa \).
2. Let \( \{L_0, \ldots, L_{n-1}\} \) be a \( (\varepsilon, \kappa) \)-realizable sequence of linear maps of length \( n \) at \( p \) and let \( \{L_n, \ldots, L_{n+m-1}\} \) be \( (\varepsilon, \kappa_2) \)-realizable of length \( m \) at \( X^n(p) \).

Then, for \( \kappa = \kappa_1 + \kappa_2 < 1 \) the sequence of linear maps \( \{L_0, \ldots, L_{n+m-1}\} \) is \( (\varepsilon, \kappa) \)-realizable.

**Proof** Item (1) follows by choosing \( Y = X \).

For item (2), given \( \varepsilon > 0 \), let \( r_1, r_2 \) be the radius according to Definition 9.27 related to the realizable linear flows \( \{L_0, \ldots, L_{n-1}\} \) and \( \{L_n, \ldots, L_{n+m-1}\} \) respectively. We take any nonempty open set \( U \subseteq B(p, r_1) \). If we have \( \mathcal{P}_X^t(p)(B(p, r_1)) \subseteq B(X^n(p), r_2) \) then we have what we need to compose and obtain the realization; otherwise we choose a smaller \( r < r_1 \).

Given \( \nu > 0 \), we decrease the radius \( r \) if necessary, by using Lemma 9.26, to get \( |\mu(K) - x(t)\mu(\mathcal{P}_X^t(p)(K))| < \nu \) for all \( t \in [0, n] \) and any measurable set \( K \subseteq B(p, r) \). By definition and choice of the radius \( r > 0 \), we have the flow-box \( \mathcal{P}_X^{n+m}(p)(U) \). Again, by definition, given any \( U \subseteq B(p, r) \) we get a measurable \( K_1 \subseteq U \) and a vector field \( Y_1 \) satisfying (a) and (b) of Definition 9.27. Also for any nonempty open subset of \( B(X^n(p), r_2) \), in particular for \( \mathcal{P}_X^n(p)(U) \), we get a measurable \( K_2 \subseteq \mathcal{P}_X^n(p)(U) \) the vector field \( Y_2 \) satisfying (a) and (b) of Definition 9.27.

Now we define the vector field \( Y = Y_1 \) in the flow-box \( \mathcal{P}_X^n(p)(U) \) and \( Y = Y_2 \) in the flow-box \( \mathcal{P}_X^{n+m}(p)(U) \) when \( Y = X \) elsewhere.

The vector field \( Y \) is \( C^1 \) since, by definition, \( (DY_1)_q = DX_q = (DY_2)_q \) for any \( q \in \mathcal{P}_X^n(p)(U) \), and so \( Y \) and \( U \) satisfies (i). To check (ii) we define \( K := K_1 \cap K_2 \) where \( K_2 \) is such that \( \mathcal{P}_X^n(p)(K_2) = K_2 \). By Lemma 9.26, we get \( x(n)\mu(\hat{U}) < \mu(K) + v \) and also \( \mu(U \setminus K_2) < x(n)\mu(\hat{U}) + v \). So we obtain

\[
\mu(U \setminus K) = \mu(U \setminus (K_1 \cap K_2)) \leq \mu(U \setminus K_1) + \mu(U \setminus K_2) < \kappa_1 \mu(U) + x(n)\mu(U \setminus K_2) + v < \kappa_1 \mu(U) + x(n)\kappa_2 \mu(U) + \nu < \kappa_1 \mu(U) + \kappa_2 \mu(U) + \kappa_2 u + \nu = \kappa \mu(U) + (1 + \kappa_2) \nu.
\]

Therefore the result follows by considering a sufficiently small \( \nu \). Finally, (ii) follows by definition.
Remark 9.29 Using elementary Vitali's covering arguments, we can show that we only have to prove realizability of the linear maps \{L_0, \ldots, L_{n-1}\} for \(U = B(p', r')\) where \(B(p', r') \subseteq B(p, r)\).

### 9.3.2.1 Small Rotations

Here we construct realizable linear flows of time-1 length at \(p\), which rotate by a small angle \(\xi\) the action of the Linear Poincaré Flow, i.e. \(L_0 := P_X^1(\hat{R}_\xi) \circ \hat{R}_\xi\) where \(\hat{R}_\xi\) is a rotation of angle \(\xi\) defined by

\[
\hat{R}_\xi := \begin{pmatrix} \cos(\xi) & -\sin(\xi) \\ \sin(\xi) & \cos(\xi) \end{pmatrix}
\]

in a canonical base of \(N_p\). Let \(\hat{R}_\xi\) be the \(3 \times 3\) matrix associated to the linear map \((x, y, z) \mapsto (x, \hat{R}_\xi y, z))\).

**Lemma 9.30** Given \(X \in X^2_\mu(M)\), a non-periodic point \(p \in M\), \(\varepsilon > 0\), \(0 < \kappa < 1\) and a fixed time \(T = 1\). Then there exists an angle \(\xi\) (not depending on \(p\) and \(\xi = O(\varepsilon)\)) such that, for every \(\gamma > 0\), there exist \(r > 0\) (depending on \(p\)) and a zero divergence vector field \(Y\), \(\varepsilon\)-\(C^1\)-close to \(X\), such that

(a) \(Y - X\) is supported in the flow-box \(\mathcal{B}^1_X(p)(B(p, r))\);

(b) \(\|P_X^1(q) - P_X^1(p)\hat{R}_\xi\| < \gamma\) for all \(q \in B(p, r \sqrt{1 - \kappa})\).

**Proof** We take a nonperiodic point \(p \in M\) and define

\[
C := \max(\|DY^1_p\| : p \in M, Y \in \Psi(X, \varepsilon)),
\]

where \(\Psi(X, \varepsilon)\) is a \(\varepsilon\)-\(C^1\)-neighborhood of \(X\). Using Lemma 9.25 we obtain a \(C^2\) conservative diffeomorphism \(\Psi: \mathcal{B} \to \mathcal{E}\) and \(T = (c, 0, 0)\). Take an uniform bound \(\Theta > 0\) for the norm of the first and second derivatives of \(\Psi\) computed in time-1 thin flow-boxes and suppose that this constant is also valid for any vector field \(\varepsilon\)-\(C^1\)-close to \(X\). We take the angle \(\xi\) such that

\[
\xi < \frac{\varepsilon(1 - \sqrt{1 - \frac{\varepsilon}{\Theta^2}})}{4\Theta^2}.
\]

(9.4)

We now fix \(\gamma > 0\). To obtain item (a), we note that for any \(\alpha < 1\) with \(\alpha \approx \varepsilon\) there exists \(r > 0\) such that \(X^{1,\alpha}(q) \cap N_{X^1(p)} = \emptyset\) for every \(q \in B(p, r)\). We fix such \(\alpha > 0\) and let \(g: \mathbb{R} \to \mathbb{R}\) be a \(C^\infty\) function such that \(g(t) = 0\) for \(t < 0\), \(g(t) = t\) for \(t \in [1 - \alpha, 2\alpha - 1]\), \(g(t) = \alpha\) for \(t \geq \alpha\) and \(\dot{g} \leq 2\).

Now for all \(q = (0, y, z) \in B(p, r)\) we decrease \(r\) so that

\[
|y|, |z| < \min \left\{ \frac{c\varepsilon}{\Theta^2 g}, \frac{\varepsilon}{2\Theta} \right\}.
\]

(9.5)
For such $r > 0$, let $G : \mathbb{R} \to [0, 1]$ be a $C^\infty$ function such that $G(\rho) = 1$ for $\rho \leq r \sqrt{1 - \frac{z^2}{2}}$, $G(\rho) = 0$ for $\rho \geq r$ and $\dot{G} \leq 2[(1 - \sqrt{1 - \frac{z^2}{2}})r]^{-1}$. We set $\rho = \sqrt{y^2 + z^2}$ and consider the rotation flow $R_{\xi G(\rho)}(0, y, z)$ acting on $N_\rho$, which we denote by $R_t(q)$, and which is defined by $R_t(0, y, z) = (0, \dot{R}_{\xi G(\rho)}(y, z))$. Denote the time derivative by $\dot{R}_t$. By a simple computation, we obtain

$$\dot{R}_t \circ R_t^{-1}(0, y, z) = \xi \dot{g}(t) G(\rho)(0, -z, y). \tag{9.6}$$

We consider the flow $T'(ct, 0, 0)$ associated to $T$ and define the map $T'(t, q) \equiv T'(R_t(q))$ for $q = (0, y, z) \in B(p, r)$. Setting $H(t, q) := (t, R_t(q))$ and

$$F(t, R_t(q)) := T'(R_t(q)) = T(t, q)$$

we obtain $T'(t, q) = F \circ H(t, q)$. Taking time derivatives at $t = s$ we obtain

$$\frac{d}{dt} T'(t, q) \bigg|_{t=s} = \frac{d}{dt} T'(R_t(q)) \bigg|_{t=s} = D F_{H(t, q)} \cdot DH_s$$

$$= (\partial_1 F \partial_2 F)_{H(t, q)} \begin{pmatrix} \partial_1 H \\ \partial_2 H \end{pmatrix} = \begin{pmatrix} T(T^s(R_{\xi})(q)) \cdot DT^s_{\dot{R}_{\xi}(q)}(q) \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}$$

The vector field $Z$ is defined in flow-box coordinates by

$$Z(\cdot) = T(\cdot) + DT^s_{\dot{R}_{\xi}(q)} \cdot \dot{R}_{\xi}(R_{\xi}(\cdot) \cdot T^{-s})(\cdot).$$

From (9.6) we deduce that the $C^1$-perturbation is given by $Z = T + P$ with

$$P(x, y, z) = \xi f(x, y, z) \cdot A(x, y, z), \tag{9.7}$$

where $f(x, y, z)$ is the scalar function $g(x/c)G(\sqrt{y^2 + z^2})$ and $A(x, y, z)$ is the linear map $(0, -z, y)$. It is straightforward to see that $\text{div}(Z) = 0$. Also, the support of the perturbation $P$ is $B(p, r) \times [0, \alpha_c) \subset \mathbb{C}$.

Now we estimate the $C^1$-norm of $P$. We note that $\|P\|_{C^1} \leq \varepsilon / \Theta$ by (9.5). To compute the $C^1$-norm we take derivatives in (9.7) and, by the product rule,

$$DP = \xi [\nabla f \cdot A(x, y, z) + f \cdot A]$$

$$= \xi \left[ \begin{array}{c} \dot{g}(x/c) G^{-1} \frac{y}{\sqrt{y^2 + z^2}} \\ \dot{g}(x/c) G^{-1} \frac{z}{\sqrt{y^2 + z^2}} \end{array} \right] \cdot A(x, y, z) + f \cdot A.$$
(\rho \cos(\beta), \rho \sin(\beta)), we get
\[
\xi \hat{\Delta} \left( \frac{x}{c} \right) \hat{\nabla} \rho \cos(\beta) \rho \sin(\beta) \leq \xi \hat{\nabla} \frac{2\rho^2}{\rho} \leq \xi \frac{4\rho}{1 - \sqrt{1 - \xi^2}} r
\leq \xi \frac{4}{(1 - \sqrt{1 - \xi^2})} \leq \frac{\epsilon}{\Theta^2}.
\]

For the other three analogous terms we proceed in the same way to obtain
\[\|DP\|_{C^0} \leq \epsilon / \Theta^2.\]
We note that we are allowed to take \(y, z\) close to zero without interfering with the size of the perturbation.
This is a key property of the \(C^1\) topology.

For \(q \in \mathcal{C}(p, r, 1 - \kappa)\) we have \(Z^l(\psi(q)) = (c, \hat{R}_t(\psi(q)))\), and so \(P^l_{X}(\psi(q)) = \hat{R}_t\) (we just have to choose \(r\) suitably small). Hence we obtain the analog of (b) for the trivial vector field.

Now it is straightforward to see that (b) follows by choosing \(r\) suitably small depending on the constants \(\gamma, C\) and \(\Theta\) which is possible by what we have seen above.

Finally, we estimate the \(C^1\)-norm of \(P_t\). Using (9.5) and the choice of \(\xi\) in (9.4) we obtain \(\|Y - X\|_{C^1} = \|P_t\|_{C^1} \leq \epsilon\), and Lemma 9.30 is proved.

Now we commute the composition with the rotation and perturb in the past.

**Lemma 9.31** Let \(X \in \mathcal{X}^2(M)\), a non-periodic point \(p \in M\), \(\epsilon > 0, 0 < \kappa < 1\), and a fixed time \(T = 1\) be given. Then there exists an angle \(\xi\) (not depending on \(p\)) such that, for every \(\gamma > 0\), there exists \(r > 0\) (depending on \(p\)) and a zero divergence vector field \(Y\), \(\epsilon\)-\(C^1\)-close to \(X\), such that:

(a) \(Y - X\) is supported in
\[\mathcal{S}^l_{X}(p)(B(p, r)) = \{[\mathcal{S}^l_{X}(p)(q), t \in [-1, 0]]\};\]

(b) \(\|P^l_{X}(q) - \hat{R}_t P^l_{X}(X^{-1}(p))\| < \gamma\) for each \(q \in \mathcal{S}^l_{X}(p)(B(p, 1 - \kappa r))\).

**Proof** We proceed as in Lemma 9.30, this time for \(X^{-l}\), finding a change of coordinates \(\hat{\psi}(x, y, z) = X^{-l}(0, \varphi(y, z))\). Then we consider \(R_0 \mathcal{S}^l_{X}(G(p))\) for \(t > 0\) and we find \(Z\). We define \(Z = \psi_a Y\) and we get
\[P^l_{X}(q) = [P^l_{Y}(Y^{-1}(q))]^{-1} \approx [P^l_{X}(p) \hat{R}_t]^{-1} = \hat{R}_t P^l_{X}(X^{-1}(p))\],
by the same arguments used in the proof of Lemma 9.30.

Now we we use the two previous lemmas to build some useful realizable linear flows.

**Lemma 9.32** Given \(X \in \mathcal{X}^2(M)\), \(\epsilon > 0, 0 < \kappa < 1\), a non-periodic point \(p\) and a fixed time \(T = 1\). Then there exists an angle \(\xi\) (not depending on \(p\)) such that \(L_0 = P^l_{X}(p) \hat{R}_t\) and \(L_0 = \hat{R}_t P^l_{X}(p)\) are \((\epsilon, \kappa)\)-realizable linear flows of length 1 at \(p\).
Proof We prove that $L_0 = P_X^1(p)\hat{R}_\xi$ is $(\varepsilon, \kappa)$-realizable. Let $\gamma > 0$. By Remark 9.29, we may choose the open set $U$ to be a ball, say $B(p', r') \subseteq B(p, r)$. Now we apply Lemma 9.30 and we get a zero divergence vector field $Y, \varepsilon, C^1$-close to $X$, such that $Y - X$ is supported inside the flow-box $\mathcal{F}_Y(p)B(p, r')$ and, for every $q \in B(p', r'\sqrt{1-\kappa})$, we have $\|P_Y^1(q) - P_X^1(p')\hat{R}_\xi\| < \gamma$. We note that, since $r > 0$ can be taken arbitrarily small, the arrival time at $N_{X^1(p)}$ for points in $B(p, r)$ is almost 1.

Taking $K = \overline{B(p', r'\sqrt{1-\kappa})} \subseteq U$, we get

$$\frac{\mu(K)}{\mu(U)} = \frac{\pi(1-\kappa)r^2}{\pi r'^2} = 1 - \kappa$$

and so the first statement of the lemma follows.

For the perturbation $F$, defined in Lemma 9.30, we have $DX_q = DY_q$ for any $q \in B(p', r') \cup \mathcal{F}_Y(p)B(p, r')$. Therefore item (i) on Definition 9.27 is true. Finally, item (ii) follows from item (b) of Lemma 9.30 and the continuity of the Linear Poincaré Flow.

For $L_0 = \hat{R}_\xi P_X^1(p)$ we proceed analogously now using Lemma 9.31. This completes the proof. \qed

**Lemma 9.33** Given $X \in X^2_\mu(M), \varepsilon > 0, 0 < \kappa < 1$ and a non-periodic point $p$, there exists an angle $\xi$ such that, for $|\xi| < \xi_i, i = 1, 2$, the composition

$$N_p \xrightarrow{P_X^1(p)} N_{X^1(p)} \xrightarrow{P_X^{1}(X^{1+i}(p))} N_{X^{1+i}(p)} \xrightarrow{\hat{R}_\xi} N_{X^{1+i}(p)}$$

is an $(\varepsilon, \kappa)$-realizable linear flow of length $r + 2$ at $p$.

**Proof** Take $\psi > 0$. By Lemma 9.32, for $\kappa < \kappa$ we get $\xi$ such that $P_X^1(p)\hat{R}_\xi$ and $\hat{R}_\xi P_X^1(X^{1+i}(p))$ are $(\varepsilon, \kappa)$-realizable. By item (1) of Lemma 9.28, the trivial map $P_X^1(p)$ is $(\varepsilon, \kappa)$-realizable. Now, if $\kappa = \kappa_1$, then we use item (2) of Lemma 9.28 and obtain the $(\varepsilon, \kappa)$-realizability. \qed

### 9.3.2.2 Large Rotations

Now we find conditions under which we can rotate by large angles. In the previous section we were able to rotate by time-1, but we need to rotate along arbitrarily long times. What then happens if we increase time?

We want to rotate by an angle $2\pi$, and thus we take a time $m$ such that $\xi m = 2\pi$. But $\xi$ is in general very small, and so $m$ must be very large. Note that the choice of $m$ may affect the norm of the perturbation because, for $\psi$ given by Lemma 9.25, $\|\psi\|$ depends on $m$ and, in general, increases with $m$. One of the reasons for this fact is that, for $v \in N_p$, we may have a very small angle $\angle(X^m(v), X(X^m(p)))$.

Furthermore, the dynamics along the orbit may also obstruct the construction of a small norm perturbation. Let us consider a situation in which this last problem is minimized, say when we have simultaneously
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(a) no domination, that is $P^\mathcal{L}_X(p)$ is “almost conformal” for all $t \in [0, m]$;
(b) almost right angles: $\angle(N^X_{x_t}(p), N^I_{x_t}(p)) \approx \frac{\pi}{2}$, for all $t \in [0, m]$.

Even if we have properties (a) and (b), our perturbations may not have a small $C^1$ norm, because the normal directions may be sent almost parallel to $E^X$. If this problem does not occur, then under conditions (a) and (b) we can perform large rotations with just a single perturbation. In fact, this was the strategy for the development of perturbations for linear differential systems in $[40, \text{Lemma 3.3}]$.

In general, we concatenate several time-1 small rotations. However, this concatenation implies that $\kappa$ grows. In $[51, \text{Lemma 3.7}]$, Bochi bypassed this problem using a nested rotation lemma. Here we adapt this method to our setting.

Let $\mathcal{E}(p) \subseteq N_p$ be an ellipse centered in $p$. As in $[51]$, the eccentricity $E$ of an ellipse is defined by $E := \sqrt{\frac{\text{major axis}}{\text{minor axis}}}$. We also consider the map $J \in SL(2, \mathbb{R})$ such that $J(\mathcal{E}(p))$ is a disk. For $r > 0$ we define the ellipse $\mathcal{E}(p, r) := J^{-1}(B(p, r))$. Given any ellipse $\mathcal{E}(p, r)$ and $J \in SL(2, \mathbb{R})$ such that $J(\mathcal{E}(p, r))$ is a disk, we denote by $\hat{E}_\xi$ the elliptical rotation defined by $J^{-1} \hat{E}_\xi J$.

**Lemma 9.34** Let there be given $X \in X^2_M(M)$, a non-periodic point $p \in M$, $\varepsilon > 0$, $0 < \kappa < 1$, a fixed time $T > 1$ and $\varepsilon \geq 1$. Then there exists $\hat{\varepsilon} > 0$ (not depending on $p$) such that, for every $\gamma > 0$, there exists $r > 0$ (depending on $p$) with the following properties.

If $\mathcal{E}(p, r)$ is an ellipse with eccentricity $E \leq \varepsilon$ and $\text{diam}(\mathcal{E}(p, r)) < \varepsilon$, $\hat{E}_\xi$ is a rotation of the ellipse $\mathcal{E}(p, r)$ satisfying $\|P^X_{\mathcal{E}_\xi}(p) - P^I_{\mathcal{E}_\xi}(p)\| < \hat{\varepsilon}$, there exists a zero divergence vector field $Y$ of class $C^1$, $\mathcal{C}^{1}$-close to $X$ such that

(a) $Y - X$ is supported in the flowbox $\mathcal{F}_{\mathcal{E}_\xi}(p)(\mathcal{E}(p, r))$;
(b) $\|P^X_{\mathcal{E}_\xi}(q) - P^I_{\mathcal{E}_\xi}(p)\| < \gamma$ for all $q \in \mathcal{E}(p, r \sqrt{1 - \kappa})$.

**Proof** The proof is the same as for Lemma 9.30, but the angle $\xi$ depends also on $E$. We use Lemma 9.25 and consider the flow-box $\mathcal{F}_{\mathcal{E}_\xi}(p)(\mathcal{E}(p, r))$. Let $J \in SL(2, \mathbb{R})$ be such that $J(\mathcal{E}(p, r))$ is a disk. Now we take the flow-box $\mathcal{F}_{\mathcal{E}_\xi}(p)(J(\mathcal{E}(p, r)))$ and define the elliptical rotation by $E_{\varepsilon\xi}(p) G(p) := J^{-1} R_{\varepsilon\xi}(p) G(p) J$. Since $E = \|J\| = \|J^{-1}\|$, $\xi$ should be smaller than the one in Lemma 9.30, which is obtained by taking $\hat{\varepsilon} \approx 0$.

The next simple lemma says that, if we fix a small ellipse in a small ball $B(p, r) \subseteq N_p$ and consider its arrival into $N^{X^1}_{X^1}(p)$, then this set is almost the image under the Linear Poincaré Flow at $p$ of the same ellipse modulo translations. A similar statement is proved in $[51, \text{Lemma 3.6}]$.

**Lemma 9.35** Let $X' : M \rightarrow M$ be a $C^1$-flow, $x \in [0, 1]$ (near 1), and $E \geq 1$. There exists $r > 0$ such that, for all ellipses $\mathcal{E}(q, r) \subseteq B(p, r) \subseteq N_p$ with eccentricity $E$, $P^X_{p}(\mathcal{E}(q, r)) - q + \mathcal{F}_{\mathcal{E}_\xi}(p)(q)$ is contained in

$$\mathcal{E}(p)(\mathcal{E}(q, r)) \subseteq P^I_{X}(p)(\mathcal{E}(q, 2(1 - \xi)r) - q) + \mathcal{F}_{\mathcal{E}_\xi}(p)(q).$$
Since $P^j_X(p)$ is modified area-preserving, we measure the non-conformality using its norm $\|P^j_X(p)\|$ in the following way: suppose that $d \geq a$ are the major axis and the minor axis (respectively) of the ellipse $E(p) = P^j_X(p)(B(p, 1))$. Then the eccentricity of $E(p)$ is $E = \sqrt{d/a}$. Since $\|P^j_X(p)\| = d$ and, by volume-preservation, we have $a^{-1} = d \cdot x(t)$, we conclude that $E = \sqrt{d/a} = \sqrt{d^2x(t)} = d\sqrt{x(t)} = \|P^j_X(p)\|\sqrt{x(t)}$.

The next lemma is a version of [51, Lemma 3.7] and says that bounded eccentricity is crucial to concatenate many elliptical small rotations keeping $\kappa$ controlled.

**Lemma 9.36**: Given $X \in \mathfrak{X}_0^1(M)$, $\varepsilon > 0$, $0 < \kappa < 1$ and $E := 1$, there exists $\delta > 0$ satisfying the following.

Let $p \in M$ be a non-periodic point and suppose that for some $n \in \mathbb{N}$ we have $\|P^j_X(p)\| \leq E \sqrt{x^{-1}(j)}$ for $j = 1, \ldots, n$. If $L_j : N_{X^j}(p) \rightarrow N_{X^{j+1}}(p)$ for $j = 0, \ldots, n - 1$ is a sequence of linear maps satisfying:

(a) $L_{j-1} \ldots L_0(B(p, 1)) = P^j_X(p)(B(p, 1))$ for $j = 1, \ldots, n$;

(b) $\|P^j_X(X^j(p)) - L_j\| < \delta$ for $j = 0, 1, \ldots, n - 1$;

then $\{L_0, L_1, \ldots, L_{n-1}\}$ is an $(\varepsilon, \kappa)$-realizable linear flow at $p$.

**Proof**: Let us start by fixing some constants. We choose $\kappa_0 < \kappa$ by taking $\lambda \in [0, 1[$ near $1$ such that $\lambda^{q_0}(1 - \kappa_0) > 1 - \kappa$. We take $\xi \in [0, 1[$ such that $\xi \in \lambda, 1$ and $2 - \xi \in \lambda^{-1}[1, \lambda^{-1}[1].$ Let $\delta$ be given by Lemma 9.34 depending on $\varepsilon$, $E$ and $\kappa_0$.

Using (a) and the hypothesis $\|P^j_X(p)\| \leq E \sqrt{x^{-1}(j)}$ (for each $j$) we consider $E_j$ the rotation of the ellipse $P^j_X(p)(B(p, 1))$. Then $L_j := P^j_X(X^j(p))E_j$ satisfies (b) and these are the linear maps which we will $(\varepsilon, \kappa)$-realize. Let $\gamma > 0$ be given.

By Lemma 9.35 applied $n$ times, and shrinking the radii at each step, there exists $r_1 > 0$ such that, for each $j$ and all ellipses $E(q, r) \subseteq B(X^j(p), r_1) \subseteq N_{X^j}(p)$ with eccentricity $E$, we have

- $P^j_X(X^j(p))(E(q, \xi r) - q) + P^j_X(X^j(p))(q) \subseteq P^j_X(X^j(p))(E(q, r_1))$ and
- $P^j_X(X^j(p))(E(q, r_1)) \subseteq P^j_X(X^j(p))(E(q, (2 - \xi) r - q) + P^j_X(X^j(p))(q))$.

Take $r < r_1$. Now we will define the vector field $Y$ and the measurable set $K$ as in Definition 9.27. By Remark 9.29, we consider $U = B(p', r') \subseteq B(p, r)$. For each $j$, denoting $P^j_X(p)(p')$ by $p_j'$, we define a sequence of ellipses $E_j' \subseteq N_{X^j}(p)$ with eccentricity $E$ by

- $E_0' = B(p', s')$ for $s \in [0, 1]$ and $E_j' = P^j_X(p)(B(p', s') - p') + p_j'$.

Decreasing $r$, if necessary, these ellipses satisfy the conditions of Lemma 9.35. Thus, for each $j$ we also have

- $E_{j+1}' \subseteq E_j' \subseteq E_j(X^j(p))(E_{j-1}')$ and $E_{j+1} \subseteq E_j \subseteq E_{j-1}'$.

For each $j$ we apply Lemma 9.34 to $p_j', \kappa_0$, $E_j' \subseteq E_j$, with $s = \lambda^{q_0}$. Hence there exists a vector field $Y_j$ such that...
9.3 \( C^1 \) Generic Incompressible Flows

(i) \( Y_j - X \) is supported in the flow-box \( \mathcal{P}_X(p_j')(\delta^j_n) \);
(ii) for every \( q_j \in \delta^j_n \sqrt{1 - \kappa_0} \) we have \( \| P_X^j(q)_j - P_X^j(p'_j) \| \leq \gamma \).

By item (i), the \( Y_j \) have disjoint supports, and so we define \( Y := \sum_{j=0}^{n-1} Y_j \). We define also \( K := \delta^j_n \sqrt{1 - \kappa_0} = \bar{B}(p'_j, \lambda^{2n} \sqrt{1 - \kappa_0} r') \). Hence we obtain

\[
\frac{\mu(K)}{\mu(U)} = \frac{\pi (\lambda^{2n} \sqrt{1 - \kappa_0} r')^2}{\pi r'^2} = \lambda^{4n} (1 - \kappa_0) > 1 - \kappa.
\]

Let us see that when we iterate we have a nested sequence, i.e., for all \( q \in K \) and each \( j \) we have \( Y^j(q) \in \delta^j_n \sqrt{1 - \kappa_0} \). We have \( \mathcal{P}_Y^j(p_j')(\delta^j_n) \subseteq \delta^j_{n+1} \subseteq \delta^j_{n+1} \), and so for every \( j \) we obtain \( \mathcal{P}_Y^j(p_j')(\delta^0_n) \subseteq \delta^j_{n-1} \subseteq \delta^j_{n-1} \). Hence for \( s = \lambda^{2n} \sqrt{1 - \kappa_0} \) we get \( \mathcal{P}_Y^j(p_j')(K) \subseteq \delta^j_{n+1} \sqrt{1 - \kappa_0} \), and the orbit of \( q \) will be inside the domain of each of these rotations.

Finally, to prove that \( \| P_X^j(Y_j(q)) - L_j \| < \gamma \) for all \( q \in K \), we use item (ii), and go back and decrease \( r \) once more, if necessary.

\[\square\]

### 9.3.3 Blending Oseledets Directions Along an Orbit Segment

Given \( p \in \Gamma_m^+(X) \), we suppose that

\[
\Delta(X^t(p), r) = \frac{\| P_X^t(X^t(p))|N^r_{X^t(p)} \|}{\| P_X^t(X^t(p))|N^r_{X^t(p)} \|} \geq c,
\]

for \( c > 1 \) and \( 0 \leq t + r \leq m \). Then the dynamics sends vectors near \( N^m_{X^t(p)} \) into vectors near \( N^m_{X^{t+r}(p)} \) during that period. The next simple lemma, whose proof may be found in [31, Lemma 3.9], clarifies this behavior: Denote two unit vectors by \( n^s \in N^r_{X^t(p)} \) for \( \sigma = u, s \).

**Lemma 9.37** Given an angle \( \xi \), there exists \( c > 1 \) such that, if \( \Delta(X^t(p), r) > c \), then there exists \( u \in N^r_{X^t(p)} \setminus \{0\} \) satisfying \( \angle(u, n^u(s)) < \xi \) and \( \angle(P_X^s(X^t(p)) \cdot u, n^{s+t}(p)) < \xi \).

The next lemma gives us the conditions under which we may apply Lemma 9.36. For a proof of the next lemma see [39, Lemma 4.0.11].

**Lemma 9.38** Let \( \xi > 0 \) and \( d > 1 \) be given. Let \( \angle(N^m_{X^t(p)}, N^m_{X^{t+r}(p)}) > \xi \) for each \( t > 0 \), and let \( d^{-1} \leq \frac{\| P_X^t(p)|N^r_{X^t(p)} \|}{\| P_X^t(p)|N^r_{X^t(p)} \|} \leq d \). Then there exists \( E > 1 \) such that \( \| P_X^t(p) \| \leq E \sqrt{x(t)} \) for all \( t > 0 \).
Now we are able to mix the Oseledets subspaces by small perturbations along orbits which have no domination.

**Lemma 9.39** Let \( X \in X^3_{\mu}(M), \varepsilon > 0 \) and \( 0 < \kappa < 1 \). There exists \( m \in \mathbb{N} \) such that, for every \( p \in \Delta(\varepsilon, \kappa) = \{ p \in \mathcal{O}^+ : \frac{||P_X^m(p)N^m_{X(p)}||}{||P_X^m(p)||^{1/2}||N^m_{X(p)}||^{1/2}} \geq \frac{1}{2} \} \), there exists an \((\varepsilon, \kappa)\)-realizable linear flow such that \( L^m(N^m_{x(p)}) = N_{X^m(p)}^m \).

**Proof** First we set up the constants. We take \( \xi > 0 \) the minimum of the angles satisfying simultaneously Lemma 9.32 and Lemma 9.33 and depending on \( X, \varepsilon \) and \( \kappa/2 \). We set \( C := \max(||DX^{-1}|| : p \in M) \) and \( c \) given by Lemma 9.37 depending on the angle \( \xi \). We take \( c > C^2 \) also.

Lemma 9.38 gives us \( E > 1 \) depending on \( \xi \) and \( d = 2c^2 \). Let \( \delta > 0 \), depending on \( X, \varepsilon, \kappa \) and \( E \), be given by Lemma 9.36. Let \( \beta > 0 \) be such that \( ||R_0 - Id|| \leq C^{-1}E^{-2}\delta \) for \( \theta = \beta \).

Finally, we take a sufficiently large \( m \in \mathbb{N} \) satisfying \( m \geq \frac{2n}{\beta} \). Now we divide the proof into three steps.

**Step I** – Small angle between the Oseledets subspaces. We assume that

\[
\text{for some } r \in [0, m] \text{ we have } \Delta(N^m_X(p), N^m_{X^r(p)}) < \xi. \tag{9.8}
\]

We take advantage of this fact and we define a realizable linear flow of length 1 in the following way. On the one hand, if \( r < m - 1 \), the linear map is based at \( X^r(p) \) and is defined by \( L_0 := P_X^1(X^r(p))R_{\xi} \). On the other hand, if \( r > m - 1 \), the linear map is based at \( X^{r-1}(p) \) and is defined by \( L_0 := R_{\xi}P_X^1(X^{r-1}(p)) \).

Now we use Lemma 9.32 and concatenate from the right and left, if necessary, with trivial maps by using item (1) of Lemma 9.28. We obtain \( L^m(N^m_{x(p)}) = N^m_{X^m(p)} \).

**Step II** – Locally \( N^r \) dominates \( N^m \). Now we assume that

\[
\text{for some } 0 \leq r + t \leq m \text{ we have } \Delta(X^t(p), r) > c. \tag{9.9}
\]

From Lemma 9.37 there exists a vector \( v \in N_{X^t(p)}^m \) such that \( \Delta(v, n_{x(p)}^m) < \xi \) and \( \Delta(P_X^s(X^t(p)) \cdot v, n_{x(p)}^m) < \xi \). Since \( \xi \) is small, we apply Lemma 9.32 at \( X^t(p) \) and at \( X^{r+t}(p) \). By the choice of \( c \) above, we get \( r > 2 \) and so we have disjoint perturbations.

Therefore, our first rotation allows us to send \( N^m_{X^r(p)} \) onto the subspace \( \mathbb{R} \cdot v \). The flow then maps this direction into \( P_X^s(X^t(p)) \cdot v \) in time \( r \) and, finally, another rotation sends \( P_X^s(X^t(p)) \cdot v \) onto \( N^m_{X^{r+t}(p)} \).

Now we use Lemma 9.28 and concatenate the three realizable linear flows, say rotation-trivial-rotation, by using Lemma 9.33 and we get \( L^m(N^m_{x(p)}) = N^m_{X^m(p)} \).

**Step III** – Conformal behavior. Finally, we suppose that we do not have either (9.8) or (9.9). We set up the conditions for Lemma 9.38. Since \( \Delta(p, m) \geq \frac{1}{8c^2} \) and (9.9) is false we have

\[
\Delta(X^r(p), r) = \Delta(X^{r+t}(p), m-t-r)^{-1} \Delta(p, m) \Delta(r, r)^{-1} \geq \frac{1}{2c^2}.\]
Therefore, since \(d = 2c^2\),

\[
\frac{1}{d} \leq \frac{\|P_{\delta}^X(X^r(p))\|_{N^r_{\delta}(1)}}{\|P_{\delta}^X(X^r(p))\|_{N^r_{\delta}(r)}} \leq d.
\]

for every \(r, t\) with \(0 \leq r + t \leq m\). We observe that, in particular, for \(r = 0\), we have \(\delta(N^r_{\delta}(1), N^r_{\delta}(p)) = \xi\) for all \(r \in [0, m]\). Now we use Lemma 9.38 and conclude that \(\|P_{\delta}^X(p)\| \leq E\sqrt{x(t)^{-1}}\) for each \(t \in [0, m]\).

Let us take \(\xi_0, \xi_1, \ldots, \xi_{m-1}\) with \(\xi_j < \beta\) for each \(j\) and satisfying also \(\sum_{j=0}^{m-1} j = \delta(N^m_{\delta}, N^m_{\delta}(p))\). We define

\[
L_j : N^j_{\delta}(p) \to N^{j+1}_{\delta}(p), \quad v \mapsto P^{j+1}_{\delta}(p)R_{\xi_j}[P^j_{\delta}(p)]^{-1}v.
\]

Let us check the conditions of Lemma 9.36. Since, by definition, for each \(j\) we have \(L_{j-1} \cdots L_0 = P^m_{\delta}(p)R_{\sum_{r=0}^{m-1} \xi_r}\), we obtain item (a) of Lemma 9.36. Now we have

\[
\|P^X_{\delta}(X^j(p)) - L_j\| \leq \|P^X_{\delta}(X^j(p)) - P^{j+1}_{\delta}(p)R_{\xi_j}[P^j_{\delta}(p)]^{-1}\|
\]

\[
= \|P^X_{\delta}(X^j(p))\|_{Id - P^j_{\delta}(p)R_{\xi_j}[P^j_{\delta}(p)]^{-1}}\|
\]

\[
\leq \|P^X_{\delta}(X^j(p))\|_{P^j_{\delta}(p)\|Id - R_{\xi_j}[P^j_{\delta}(p)]^{-1}\|\|\|}
\]

\[
\leq \|P^X_{\delta}(X^j(p))\|_{P^j_{\delta}(p)\|\|\|\|Id - R_{\xi_j}\|}
\]

\[
\leq CE\sqrt{x^{-1}(j)E\sqrt{x(j)}}\|Id - R_{\xi_j}\|.
\]

(In the last inequality we have used \(\|P^{-1}_{\delta}\| \leq E\sqrt{x(t)}\).) Therefore we obtain

\[
\|P^X_{\delta}(X^j(p)) - L_j\| \leq CE^2\|Id - R_{\xi_j}\| \leq \delta
\]

and item (b) of Lemma 9.36 is true. From this lemma we have the realizability, and therefore

\[
L^m(N^m_{\delta}) = L_{m-1} \circ \cdots \circ L_0(N^m_{\delta}) = P^m_{\delta}(p)R_{\sum_{r=0}^{m-1} \xi_r}(N^m_{\delta}) = P^m_{\delta}(p) : N^m_{\delta} = N^m_{\delta}(p),
\]

which proves the lemma. \(\square\)

### 9.3.4 Lowering the Norm: Local Procedure

We begin this section by adapting to our setting the results [51, Lemma 3.12] and [53, Lemma 4.5]. The first lemma (for a proof see [39, Lemma 5.0.13]) gives us information about when we have a recurrence to a positive measure set. The second lemma is an elementary result which relates the original norm to a new norm.
Lemma 9.40 Let $X^t : M \to M$ be a measurable $\mu$-invariant flow, $\Delta \subseteq M$ a positive measure set, $\Gamma := \bigcup_{t \in \mathbb{R}} X^t(\Delta)$ and let $\gamma > 0$.

Then there exists a measurable function $T : \Gamma \to \mathbb{R}$ such that, for $\mu$-a.e. $p \in \Gamma$, all $t \geq T(p)$ and every $\tau \in [0, 1]$, there exists some $s \in [0, 1]$ satisfying $|\frac{t}{\gamma} - \tau| < \gamma$ and $X^s(p) \in \Delta$.

Consider $p, q := X^t(p) \in \Gamma$ and the map $P : N_p \to N_q$ whose matrix written with respect to the Oseledets basis (given by $\{n^u_p, n^s_p\}$ and $\{n^u_q, n^s_q\}$) is

$$P = \begin{pmatrix} a^{uu} & a^{us} \\ a^{su} & a^{ss} \end{pmatrix}. $$

Let $\|P\|_{\text{max}} = \max\{|a^{uu}|, |a^{us}|, |a^{su}|, |a^{ss}|\}$.

Lemma 9.41 We have the bounds

(a) $\|P\| \leq 4(\sin \angle(N^u_p, N^u_p))^{-1}\|P\|_{\text{max}}$;

(b) $\|P\|_{\text{max}} \leq (\sin \angle(N^s_q, N^s_q))^{-1}\|P\|$.

Now we are able to decrease the norm under a small perturbation.

Lemma 9.42 Let $X \in \mathcal{X}^2(M)$, where $X^t$ is aperiodic and all hyperbolic sets have zero measure. Let $\epsilon, \delta > 0$, $0 < \kappa < 1$. Then there exists a measurable function $T : M \to \mathbb{R}$ such that for $\mu$-a.e. $p \in M$ and every $t \geq T(p)$, there exists a $(\epsilon, \kappa)$-realizable linear flow at $p$ with length $t$ such that $\|L^t(p)\| \leq \epsilon \delta$.

Proof First we take $m \in \mathbb{R}$ large enough given by Lemma 9.39 and depending on $X, \epsilon, \kappa, \delta$. Write $\Gamma := \Gamma^m(\Delta)$ and note that $X^t$ aperiodic implies that $\mu(\Gamma^m(X)) = \mu(\Gamma)$. We have $\mu(\Gamma) > 0$ and $\mu(\Gamma) = 0$, $\mu$-a.e. point $p \in M$ would be such that $\lambda^+(p) = 0$ and there is nothing to prove, because a trivial map does the work.

We recall that $\Gamma = \bigcup_{t \in \mathbb{R}} X^t(\Delta)$. For $\mu$-a.e. $p \in \Gamma$, the Oseledets' Theorem in the particular case of a three-dimensional conservative flow gives us $Q(p)$ such that for all $t \geq Q(p)$

1. $\frac{1}{t} \log \|P_X^t(p) \cdot n^u\| < \lambda^+(p) + \delta$ for all $n^u \in N^u(p) \setminus \{0\}$;
2. $\frac{1}{t} \log \|P_X^t(p) \cdot n^s\| < -\lambda^+(p) + \delta$ for all $n^s \in N^s(p) \setminus \{0\}$;
3. $-\log \sin \angle(N^u_X(p), N^s_X(p)) < t \delta$.

By using Lemma 9.40, with $\tau = 1/2$, we get recurrence to $\Delta$ approximately in the middle of the orbit segment. However, to get good estimates of the norm of the linear map $L'$, points in the orbit after this time must also satisfy items (1) and (2) above.

Let $B_n := \{p \in \Gamma : Q(p) \leq n\}$ for $n \in \mathbb{N}$. We have $B_n \subseteq B_{n+1}$ and $\mu(\Gamma \setminus B_n) \to 0$. 


We consider now the following family of sets:

\[ C_0 := \emptyset \quad \text{and} \quad C_n := \bigcup_{\epsilon \in \mathbb{R}} X^\epsilon(\Delta \cap X^{-m}(B_n)). \]

It is easy to see that \( C_n \xrightarrow{n \to \infty} \Gamma \), and so the measurable function \( T : \Gamma \to \mathbb{R} \) will be \( \mu \)-a.e. defined on each \( C_n \setminus C_{n-1} \) for \( n \in \mathbb{N} \).

Taking \( c > \max \{ \log \| D^X \lambda_p \| : p \in M \} \) yields the Lyapunov exponents of any \( p \in \mathcal{O} \) less than \( c \) (recall the definition of \( \mathcal{O} \) at the beginning of Sect. 9.3). For \( p \in \Gamma \) we have non-zero Lyapunov exponents, and so we have well-defined Oseledets one-dimensional subspaces \( N^s_p \) and \( N^u_p \).

Let \( \gamma = \min \{ 1/6, \delta/c \} \). Now we use Lemma 9.40, replacing \( \Delta \) by \( \Delta \cap X^{-m}(B_n) \) and \( \Gamma \) by \( \bigcup_{\epsilon \in \mathbb{R}} X^\epsilon(\Delta \cap X^{-m}(B_n)) \). By this lemma, for each \( n \), there exists a measurable function \( T_n : C_n \to \mathbb{R} \) such that, for \( \mu \)-a.e \( p \in C_n \) and for all \( t \geq T_n(p) \), there exists some \( s \in [0, t] \) satisfying \( X^s(p) \in \Delta \cap X^{-m}(B_n) \) and \( |s - \frac{t}{2}| < \gamma \).

Now we define a sufficiently large \( T(p) \) for \( p \in C_n \setminus C_{n-1} \) so that

\[
T(p) \geq \max \left\{ T_n(p), \frac{m}{\gamma}, 6Q(p), \frac{1}{\delta} \log \frac{4}{\sin \angle(N^s_p, N^u_p)} \right\}
\]

(9.10)

Let \( p \in C_n \setminus C_{n-1} \) and \( t \geq T(p) \). Since \( t \geq T(p) \geq T_n(p) \), we obtain \( X^t(p) \in \Delta \). Hence, by Lemma 9.39, we define a \((\epsilon, \kappa/2)\)-realizable linear flow \( L_1 : N^s_{X^t(p)} \to N^s_{X^{t+\kappa}(p)} \) sending \( N^s_{X^t(p)} \) into \( N^s_{X^{t+\kappa}(p)} \). Now we concatenate from right to left with trivial maps and, by Lemma 9.28, we obtain a \((\epsilon, \kappa)\)-realizable linear flow defined by

\[
N^s_p \xrightarrow{L_0} N^s_{X^t(p)} \xrightarrow{L_1} N^s_{X^{t+\kappa}(p)} \xrightarrow{L_2} N^s_{X^t(p)}
\]

with \( L_0 = P^s_{X^t(p)} \) and \( L_2 = P^s_{X^{t+\kappa}(p)}(X^{t+\kappa}(p)) \).

To estimate \( \| L'(p) \| \) we consider the linear maps relative to a suitable unitary basis \( [n^s_{X^t(p)}, n^s_{X^t(p)}] \) for \( r \in [0, t] \), which is invariant for the Linear Poincaré Flow, so they have the form

\[
L_2 = \begin{pmatrix}
a_{uu} & 0 \\ 0 & c^{ss}
\end{pmatrix}, \quad L_1 = \begin{pmatrix} b_{uu} & b_{us} \\ b_{us} & b_{ss}
\end{pmatrix}, \quad L_0 = \begin{pmatrix} a_{uu} & 0 \\ 0 & a^{ss}
\end{pmatrix}.
\]

The key observation is that \( b_{uu} = 0 \). Consider the product matrix

\[
L'(p) = L_2 \cdot L_1 \cdot L_0 = \begin{pmatrix} 0 & a_{uu}b_{us}c^{ss} \\ a^{ss}b_{uu}c_{uu} & a^{ss}b_{ss}c_{ss}
\end{pmatrix}.
\]

Claim: For \( p \in C_n \setminus C_{n-1} \) and \( t \geq T(p) \) we have:

(a) \( \max \{ \log |a_{uu}|, \log |c^{ss}| \} < \frac{1}{\gamma} (\lambda^+(p) + 4\delta) \);

(b) \( \max \{ \log |a^{ss}|, \log |c^{ss}| \} < \frac{1}{\gamma} (-\lambda^+(p) + 4\delta) \).

Proof of the claim: To prove that \( \log |a_{uu}| < \frac{1}{\gamma} (\lambda^+(p) + 4\delta) \) we first note that

\[
s > t(1/2 - \gamma) > t/3 \geq T(p)/3 \geq Q(p) \]

and so, by Oseledets' Theorem, we have...
\[
\log|a^{uu}| = \log|P^u_\lambda(p) \cdot n^u_p| < s(\lambda^+(p) + \delta). \text{ Since } \gamma \lambda^+(p) < \gamma c \leq \delta \text{ and } \gamma < 1/2, \text{ we get } \\
\quad s(\lambda^+(p) + \delta) < t(1/2 + \gamma)(\lambda^+(p) + \delta) < t(\lambda^+(p)/2 + \delta/2 + \lambda^+(p)\gamma + \gamma\delta) \quad < t(\lambda^+(p)/2 + \delta/2 + \lambda^+(p)/2 + \delta/2) < \frac{1}{2} t(\lambda^+(p) + 4\delta)
\]
and the inequality follows.

To prove that \( \log|e^{uu}| < \frac{1}{2} t(\lambda^+(p) + 4\delta) \) we consider the fact that \( X^t(p) \in X^{-m}(B_n) \), therefore \( X^{s+m}(p) \in B_m \) and \( Q(X^{s+m}(p)) \leq n \) by definition of \( B_n \). So we will have the approximation rate given by Oseledet’s Theorem if \( t - m - s > n \). By (9.10) for \( t \geq T(p) \), we have \(-m/t \geq -\gamma\). Since \(-s/t > -1/2 - \gamma\) and \(-\gamma \geq -1/6 \) we obtain

\[
t - m - s = t \left( 1 - \frac{m}{t} - \frac{s}{t} \right) > t \left( \frac{1}{2} - 2\gamma \right) > \frac{t}{6} \geq Q(p) \geq n.
\]
Thus \( t - m - s \) will be sufficiently large to use item (1) above. Hence

\[
\log|e^{uu}| = \log|P^{t-m-s}_X(X^{s+m}(p)) \cdot n^{uu}_X(p)| < (t - m - s)(\lambda^+(p) + \delta) \\
< t(1 - m/t - s/t)(\lambda^+(p) + \delta) < t(\gamma/1 + 2\delta)(\lambda^+(p) + \delta) \\
= t(\gamma \lambda^+(p) + \gamma \delta + \lambda^+(p)/2 + \delta/2) \\
< t(\delta + \delta/2 + \lambda^+(p)/2 + \delta/2) = \frac{1}{2} t(\lambda^+(p) + 4\delta).
\]

We note that item (b) is analogous to item (a) and the claim is proved.

Now we estimate \( \|L_1\|_{\text{max}} \). First note that

\[
s + m > t(1/2 - \gamma + m/t) > t(1/2 - \gamma) > t/6 > Q(p) \geq n,
\]
and so by (3) we have \( (\sin\angle(N^{s+m}_X(p), N^s_{X^{s+m}(p)})^{-1} < e^{(s+m)\delta} < e^{\delta}. \) Since \( L_1 \) is \((\epsilon, \kappa)\)-realizable, we conclude that \( \|L_1 - P^u_\lambda(X^t(p))\| \) is small. Therefore, because \( t > T(p) \geq m/\gamma \) and \( \gamma c \leq \delta \), we get \( \|L_1\| \leq e^{mc} \leq e^{\gamma c} \leq e^{\delta}. \) By Lemma 9.41(b) we also have

\[
\|L_1\|_{\text{max}} \leq \sin^{-1}(N^{s}_X(p), N^{s}_X(p)) \|L_1\| \leq e^{2\delta}.
\]

Now we estimate each of the entries of the product matrix:

\[
|a^{uu}b^{uu}c^{uu}| \leq e^t(\lambda^+(p) + 4\delta) + 2\delta + 1/2(\lambda^+(p) + 4\delta) = e^{6\delta}; \\
|a^{uu}b^{uu}c^{uu}| \leq e^t(\lambda^+(p) + 4\delta) + 2\delta + 1/2(\lambda^+(p) + 4\delta) = e^{6\delta}; \\
|a^{uu}b^{uu}c^{uu}| \leq e^t(\lambda^+(p) + 4\delta) + 2\delta + 1/2(\lambda^+(p) + 4\delta) \leq e^{-1/2(\lambda^+(p) + 6\delta) \leq e^{6\delta}}.
\]
This implies the inequality $\|L'(p)\|_{\text{max}} < e^{\delta \delta}$. By item (a) of Lemma 9.41 we have

$$\|L'(p)\| \leq \frac{4}{\sin \angle(N^\perp_p, N^\perp_p)} \|L'(p)\|_{\text{max}}.$$ 

But $t \geq T(p) \geq \frac{1}{4} \log \frac{4}{\sin \angle(N^\perp_p, N^\perp_p)}$ so $\frac{4}{\sin \angle(N^\perp_p, N^\perp_p)} \leq e^{\delta \delta}$ and we get $\|L'(p)\| \leq e^{\delta \delta}$. Replacing $\delta$ by $\delta/7$ we conclude that $\|L'(p)\| \leq e^{\delta \delta}$ and the lemma is proved. \(\square\)

### 9.3.4.1 Realizing Vector Fields

Let $X \in \mathfrak{X}^2_0(M)^*$, where $X^t$ is aperiodic and also all hyperbolic sets have zero Lebesgue measure. Given $\epsilon, \delta > 0$ and $0 < \kappa < 1$, we assume that $m$ is large enough to satisfy Lemma 9.39. By Lemma 9.42, there exists a measurable function $T : M \to \mathbb{R}$ such that, for $\mu$-a.e. $p \in M$ and for every $t \geq T(p)$, there exists a $(\epsilon, \kappa)$-realizable linear flow at $p$ with length $t$ such that $\|L'(p)\| \leq e^{\delta \delta}$.

This means that, for each $\gamma > 0$, we can find $r = r(p, \gamma) > 0$ such that, for every open subset $U$ of $B(p, r)$, there exists a conservative vector field $Y$, $\epsilon$-$C^1$-close to $X$, satisfying the following. First, $Y = X$ outside the flow-box $\mathcal{F}_X^\gamma(p)(U)$ and, moreover, $\|P_Y(q) - L'(p)\| \leq \gamma$ for $q$ in a measurable subset $K \subset U$ which is close to $U$ in measure (i.e. $\bar{\mu}(K) > (1 - \kappa)\bar{\mu}(U)$ for a small $\kappa > 0$); see Definition 9.27.

Since $\|L'(p)\| \leq e^{\delta \delta}$ we conclude that $\|P_Y(q)\| \leq e^{\delta \delta} + \gamma$ for all $q \in K$ (where $\gamma \approx 0$). The vector field $Y$ realizes the abstract map $L'(p)$, i.e., $P_Y(q) \approx L'(p)$ for a large percentage of points $q \in U \subset N_p$ (see Fig. 9.5).

### 9.3.5 Lowering the Norm: Global Procedure

Now we use the local construction of realizable linear flows with a small norm in order to decrease the function $LE(\cdot)$ for a zero divergence vector field $Y$ $C^1$-close
to \( X \). We will use the notion of suspension or special flow built under a function, from Sect. 2.3.2.2.

It follows from the Ambrose-Kakutani Theorem [12, 13] that any aperiodic flow is isomorphic to some special flow. This theorem is applicable to the flow of Proposition 9.23. The isomorphism is given by \( W : M \to M_h \), where \( M_h \) is the flow under the roof function \( h \). The measure \( \mu^* = W_* \mu \) is decomposed into the product of Lebesgue measure in \( \mathbb{R} \) and an \( R \)-invariant measure \( \tilde{\mu} \) in the base \( \Sigma \), that is, \( \int_{M_h} f(x, s) d\mu^* = \int_{\Sigma} \left( \int_0^{h(x)} f(x, s) ds \right) d\tilde{\mu}(x) \). Thus we have a simplified representation of our flow \( \tilde{X}^i(p) \). Hereafter we assume that our flow has this representation.

### 9.3.5.1 Sections of Flows and Special Flows

Given a special flow over a section \( \Sigma \), the set \( Q = \bigcup_{i \in \mathbb{R}} X^i(\Sigma) \) is called the Kakutani castle. The tower of height \( i \), which is denoted by \( T_i \), is the set below the graph of \( h(B_i) \), where \( B_i = \{ x \in \Sigma : h(x) = i \} \), that is, \( T_i = X^{[0,1]}(B_i) \). The next lemma is the flow version of [51, Lemma 4.1].

**Lemma 9.43** Let \( X^i : M \to M \) be a \( \mu \)-preserving aperiodic flow. For every positive measure set \( U \subseteq M \) and every \( h \in \mathbb{R} \), there exists a \( \mu \)-positive measure section \( B \subseteq U \) such that \( X^{[0,h]}(B) \) is a flow-box and \( B \) is maximal (i.e. no set containing \( B \) and with larger measure has the same properties as \( B \)).

**Proof** See [39, Lemma 6.1.2]. \( \square \)

For a \( \mu \)-generic point \( p \), Lemma 9.42 gives us \( T(p) \) which, in general, is very large. Hence Lemma 9.43 is crucial to avoid the overlapping of perturbations.

Consider a vector field \( X \) under the conditions of Proposition 9.23. For all \( Y \in C^1 \) close to \( X \) we define \( C := \max(\| P_t^X(p) \| : p \in M) \). We take \( \kappa = \delta^2 \). Using the function given by Lemma 9.42, we define \( Z_h := \{ p \in M : T(p) \leq h \} \). Then we get \( \mu(M \setminus Z_h) \to 0 \), and so for \( h \) sufficiently large we have

\[
\mu(M \setminus Z_h) < \delta^2. \tag{9.11}
\]

We intend to build a special flow with a ceiling function with height not less than \( h \) and section inside \( Z_h \). Since, for large \( h \), the set \( Z_h \) has almost full measure, by Lemma 9.43, we get a \( \mu \)-positive measure set \( B \subseteq Z_h \). If \( x \in B \), then \( h(x) \geq h \) and, since \( B \subseteq Z_h \), we have \( h \geq T(x) \), and so the conditions of Lemma 9.42 are satisfied.

Let \( \hat{Q} \) be the castle with base \( B \). We have \( \hat{Q} \supseteq Z_h \) in the measure theoretical sense so, by (9.11), we get the inequality \( \mu(\hat{Q}^i) \leq \delta^2 \). We define the subcastle \( Q \subseteq \hat{Q} \) by excluding the towers of \( \hat{Q} \) with height larger than \( 3h \). Adapting [51, Lemma 4.2] (the details are fully presented in [39, Lemma 6.2.1]) we obtain

\[
\mu(\hat{Q} \setminus Q) < 3\delta^2. \tag{9.12}
\]
9.3.5.2 The Zero Divergence Vector Field $Y$, $C^1$-Close to $X$

Now we make use of the realizability of vector fields and the properties of special flows to construct a conservative vector field $Y$ inside the subcastle $Q$ by gluing a finite number of local perturbations supported on self-disjoint flow-boxes. We note that the measures $\mu$ and $\overline{\mu}$ are equivalent. In the next lemma we follow [53, Lemma 4.14].

**Lemma 9.44** Given $\gamma > 0$, there exist $Y$, $C^1$-close to $X$, a castle $U$ for $Y'$ and a subcastle $K$ for $Y'$ such that:

(a) the castle $U$ is open;
(b) $\mu(U \setminus Q) < \gamma$ and $\mu(Q \setminus U) < \gamma$;
(c) $\mu(U \setminus K) < \kappa(1 + \gamma)$;
(d) $Y'(U) = X'(U)$ and $Y' = X'$ outside the castle $U$;
(e) if $q$ is in the base of $K$ and $h(q)$ is the height of the tower of $K$ that contains $q$, then

$$\|Y_{\gamma}^{h(q)}(q)\| \leq e^{\delta h(q)} + \gamma.$$

**Proof** The castle $Q$ is a measurable set and, since $\mu$ is Borel regular, there exists a compact $J \subseteq Q$ such that

$$\mu(Q \setminus J) < \gamma \mu(\hat{Q})/2. \quad (9.13)$$

The compact $J$ is a $X'$-castle with the same structure as $Q$ (i.e., preserving the same dynamics of bases and towers as is the case for the castle $Q$). Now we choose an open castle $V$ such that $J \subseteq V$ with

$$\mu(V \setminus J) < \gamma \mu(\hat{Q}). \quad (9.14)$$

and also with the same structure of $Q$ and $J$. For every point $p_1$ in $J \cap B$ we have $h(p_1) \geq h$. Since $(J \cap B) \subseteq Z_h$ we have $T(p_1) \leq h$, and therefore $T(p_1) \leq h \leq h(p_1)$. So, for all $t_1 \geq T(p_1)$, and for $\gamma$ fixed, there exists a radius $r_1(p_1, t_1)$ (decrease $r_1$ if one leaves the open castle $V$) such that for almost (related with $\kappa = \delta^2$) every point in $U_1 = B(p_1, r_1) \subseteq N_{p_1}$, more precisely for each point $q \in K_1 \subseteq U_1$, we have a vector field $Y_1$ supported in a small flow-box containing the orbit segment $X^{[0,t_1]}(p_1)$ such that $\|Y_1(q)\| \leq e^{\delta t_1} + \gamma$.

We continue by choosing $p_i$'s and by Vitali's arguments we fill up $J$ with a union $U$ of self-disjoint open flow-boxes in order to obtain

$$\mu(J \setminus U) \leq \gamma \mu(J)/2. \quad (9.15)$$

The set $U$ is a $X'$-castle with its section (base of the castle) the union of the $U_i$. So for each $i$ we get a vector field $Y_i$ supported in a small flow-box containing the orbit segment $X^{[0,t_1]}(p_i)$, $C^1$-close to $X$, and such that $\|Y_i(q)\| \leq e^{\delta t_1} + \gamma$ for all $q \in K_1 \subseteq U_1$. 


We define $Y = Y_t$ inside each flow-box and $Y = X$ outside. Since these flow-boxes are pairwise disjoint, the vector field is well defined and it is $\varepsilon \cdot C^1$-close to $X$. Note that $V$ is also a castle for $Y_t$, and $U$ is also a $Y_t$-subcastle of the $Y_t$-castle $V$ (having for base the union of all $U_i$). We take $K$ as the $Y_t$-subcastle with a section equal to the union of all the $K_i$. By construction of $U$, we get items (a), (d) and (e).

Now we prove item (b). Recalling that $V \supseteq U$ and $J \subseteq Q$, by (9.14) we obtain

$$\mu(U \setminus Q) < \mu(V \setminus J) < \gamma \mu(\hat{Q}) \leq \gamma.$$  

To prove that $\mu(Q \setminus U) < \gamma$ we use (9.13) and (9.15) and conclude that

$$\mu(Q \setminus U) \leq \mu(Q \setminus J) + \mu(J \setminus U) < \gamma \mu(\hat{Q}) < \gamma.$$  

Finally, for item (c) we observe that from item (b) we deduce that $\mu(U) < \mu(Q) + \mu(U \setminus Q) < 1 + \gamma$. The inequality $\mu(U \setminus K) < \kappa \mu(U)$ then leads to $\mu(U \setminus K) < \kappa(1 + \gamma)$. The proof is complete. 

\[9.3.5.3\text{ Computing } LE(Y)\]

We take $t = h\delta^{-1}$ (we may assume that this is an integer). By Lemma 9.22, we obtain $LE(Y) \leq \int_{M} \frac{1}{t} \log \|P^t_Y(p)\|d\mu(p)$. By the above construction for orbit segments inside the castle $K$ and starting in the base, we guarantee a small upper Lyapunov exponent. Therefore we define the set of points whose orbit stays for a long time in $K$ by $G := \{p \in M : Y_t(p) \in K \forall s \in [0, t]\}$ and denote by $G^c$ its complementary set.

**Lemma 9.45** For $p \in G$ we have $\|P^t_Y(p)\| < e^{t(1+6 \log C)\delta}$ for some $C > 0$.

**Proof** Let $p \in G$. We split the orbit segment $X^{[0,1]}_p$ by return-times at $B_K$ (the section of the castle $K$), say $t = b + r_1 + \cdots + r_2 + r_1 + a$, where

$$X^a(p), X^{a+\delta}(p), X^{2a+\delta}(p), \ldots, X^{\sum_{i=1}^n \delta}(p)$$

are all in the base $B_K$. It is clear that $a, b, r_i \in [0, 3\delta]$ except when $p \in B_K (a = 0)$ and $X^1(p) \in B_K (b = 0)$. Note that

$$\|P^t_Y(p)\| = \|P^b_X(p)\| \leq \|P^b_X(X^{\sum_{i=1}^n \delta}(p))\| \times \|P^r_X(X^{\sum_{i=1}^n \delta}(p))\| \times \cdots \times \|P^a_X(X^a(p))\| \times \|P^a_X(X^a(p))\|$$

But these maps are based at points in $B_K$. Hence by Lemma 9.48(e) and for $C := \max\{\|DX^1_X\| : p \in M\}$ (this constant is valid for any vector field $\varepsilon \cdot C^1$-close to $X$) we get

$$\|P^t_Y(p)\| \leq C^{3\delta} e^{(b+\sum_{i=1}^n \delta)\delta} C^{3\delta} \leq e^{(b+\sum_{i=1}^n \delta)\delta} C^{6\delta} \leq e^{t(1+6 \log C)\delta}$$

concluding the proof.
For points in $G^c$ we use inequality (9.12), Lemma 9.44 and some elementary observations (see [53, Lemma 4.16]) to deduce the following.

**Lemma 9.46** Let $\gamma = \delta h^{-1}$ as in Lemma 9.44. Then $\mu(U \cup \Gamma_m^+(X) \setminus G) < 12\delta$.

Now using Lemma 9.46 we obtain $\mu(G^c) < 12\delta$. Finally, we finish the proof of Proposition 9.23 and consequently Theorem 9.3:

\[
LE(Y) = \inf_{n \geq 1} \int_M \frac{1}{n} \log \| P_t^Y(p) \| d\mu(p) \leq \int_M \frac{1}{t} \log \| P_t^Y(p) \| d\mu(p) \\
\leq \int_G \frac{1}{t} \log \| P_t^Y(p) \| d\mu(p) + \int_{G^c} \frac{1}{t} \log \| P_t^Y(p) \| d\mu(p) \\
\leq (1 + 6 \log C) \delta \mu(G) + \log C \mu(G^c) = (1 + 18 \log C) \delta.
\]

Now we replace $\delta$ by $\frac{\delta}{(1 + 18 \log C)}$ in the proof, and Proposition 9.23 is complete.

### 9.3.6 Proof of the Dichotomy with Singularities (Theorem 9.4)

We begin by noting that, since $X'$ is aperiodic, the measure of all singularities is zero. Moreover, when we estimate the $C^1$-norm of the perturbation $P$, defined in (9.7), the choice of $r(p)$ in (9.5) guarantees that even near singularities (when $c \approx 0$), the perturbation can be done. Furthermore, if in Theorem 9.24 we take $\Omega$ with $C^{\infty}$ boundary and $g, f$ also $C^{\infty}$, the diffeomorphism $\varphi$, provided by Dacorogna-Moser, is also $C^{\infty}$. So our conservative flow-box theorem guarantees a $C^{\infty}$ conservative change of coordinates $\Psi$. Note that the perturbation $P$, defined in (9.7), is also $C^{\infty}$; moreover we know by [277] that $\mathcal{X}_{C}^{\infty}(M)$ is $C^1$-dense in $\mathcal{X}_{C}^{\infty}(M)$.

#### 9.3.6.1 Adapting the Proof with Singularities, from Theorem 9.3

The following proposition is similar to Proposition 9.23. The main difference is where the computation of the entropy function is done.

**Proposition 9.47** Let $X \in \mathcal{X}_{C}^{\infty}(M)$ and $\varepsilon, \delta > 0$. There exists $m \in \mathbb{N}$ and a zero divergence $C^{\infty}$ vector field $Y$, $\varepsilon$-$C^1$-close to $X$, which equals $X$ outside the open set $\Gamma_m(X)$ and is such that $LE(Y, \Gamma_m(X)) < \delta$.

**Proof** For a fixed $m \in \mathbb{N}$ we have $p \in \Gamma_m^+(X) \setminus \Gamma_m^*(X)$ if $p$ is periodic, has positive Lyapunov exponent and belongs to $\Gamma_m(X)$. We consider the following simple claim. For a proof see [40, Lemma 3.1].

**Lemma 9.48** Given $\delta > 0$, there exists $m \in \mathbb{N}$ such that $\mu(\Gamma_m^+(X) \setminus \Gamma_m^*(X)) < \delta$. 
We take \( m \in \mathbb{N} \) satisfying both Lemma 9.39 and Lemma 9.48 and let \( T : \Gamma^s_m(X) \to \mathbb{R} \) be similar to the function of Lemma 9.42. We now define
\[
Z_h = \{ p \in \Gamma^s_m(X) : T(p) \leq h \}.
\]
It is standard that \( \mu(\Gamma^s_m(X) \setminus Z_h) \xrightarrow{h \to \infty} 0 \), and so we may choose \( h > 0 \) satisfying
\[
\mu(\Gamma^s_m(X) \setminus Z_h) < \delta^2 \mu(\Gamma^s_m(X)).
\]
Now we increase \( h \), if necessary, and use Oseledets’ Theorem to obtain the inequality
\[
\| P^t_X(p) \| < e^{t \delta} \quad \text{for all } t \geq h,
\]
for \( p \in \Theta^0(X) \). Since \( X^t : \Gamma^s_m(X) \to \Gamma^s_m(X) \) is an aperiodic flow, we can follow the construction of section 9.3.5.2 and finally compute \( LE(Y, \Gamma^s_m(X)) \). Analogously, we define
\[
G := \{ p \in \Gamma^s_m(X) : Y^s(p) \in K, \quad \text{for all } s \in [0, t] \},
\]
and denote \( \frac{1}{t} \log \| P^{t}_X(p) \| \) by \( A^t_p(\cdot) \). Then we compute
\[
LE(Y, \Gamma^s_m(X)) \leq \int_{\Gamma^s_m(X)} A^t_p(Y) \, d\mu
\]
\[
\leq \int_{\Gamma^s_m(X) \setminus (U \cup \Gamma^+_{\Delta} m(X))} A^t_p(Y) \, d\mu + \int_{U \cup \Gamma^+_{\Delta} m(X) \setminus G} A^t_p(Y) \, d\mu
\]
\[
+ \int_{G} A^t_p(Y) \, d\mu.
\]
Since \( Y = X \) outside \( U \), by (9.16) we obtain
\[
\int_{\Gamma^s_m(X) \setminus (U \cup \Gamma^+_{\Delta} m(X))} A^t_p(Y) \, d\mu \leq \int_{\Gamma^s_m(X) \setminus \Gamma^+_{\Delta} m(X)} A^t_p(Y) \, d\mu \leq \delta.
\]
Setting \( C := \max\{\| P^{t}_X(p) \| : p \in M\} \) and using Lemma 9.48 and Lemma 9.46, we conclude that \( \int_{U \cup \Gamma^+_{\Delta} m(X) \setminus G} A^t_p(Y) \, d\mu \leq 13\delta \log C \). Finally, on \( G \), our construction allows us to obtain \( \int_{G} A^t_p(Y) \, d\mu(p) \leq \delta \) and the proposition is proved.

### 9.3.6.2 The Concluding Argument

Let \( \tilde{X} \in \mathbb{X}^1_{\mu}(M) \) and \( \tilde{\varepsilon} > 0 \) be given. We will prove that there exists \( Y \in \mathbb{X}^1_{\mu}(M) \), \( \tilde{\varepsilon} \)-\( C^1 \)-close to \( X \) satisfying the conclusions of Theorem 9.4. For \( \varepsilon = \tilde{\varepsilon}/2 \), there exists \( X \in \mathbb{X}^{\infty}_{\mu}(M) \) \( \varepsilon \)-\( C^1 \)-close to \( \tilde{X} \). It suffices to prove Theorem 9.4 for the vector field \( X \) and \( \varepsilon > 0 \).
Proof of Theorem 9.4 Let $X \in \mathcal{X}_{\mu}^{\infty}(M)$ and $\varepsilon > 0$. We will find $Y \varepsilon$-C$^1$-close to $X$ and a partition $M = D \cup O$ into $Y^\varepsilon$-invariant sets such that $\forall p \in O$ we have zero Lyapunov exponents and $D$ is a countable increasing union of compact invariant sets $A_{m}$, admitting an $m_{n}$-dominated splitting for the Linear Poincaré Flow. We define the sequence $\{X_{n}\}_{n \geq 0} \in \mathcal{X}_{\mu}^{\infty}(M)$, $m_{n} \in \mathbb{N}$, and eventually $\varepsilon_{n} > 0$ for $n \geq 0$.

Take $X_{0} = X$, $\theta > 1$ (near 1) and $\delta_{n} \to 0$.

If $\int_{\Gamma_{m}(X)} \lambda^{+}(X) d\mu = 0$ for some $m \in \mathbb{N}$, then we are finished by taking $Y = X$, $D = A_{m}(X)$ and $O$ a full measure subset of $\Gamma_{m}(X)$. Otherwise, for some $m = m_{0}$ and $X = X_{0}$, we have $\int_{\Gamma_{m_{0}}(X_{0})} \lambda^{+}(X_{0}) d\mu > 0$. Let $\epsilon_{0} \in (0, \varepsilon/2)$ be such that

$$\int_{\Gamma_{m_{0}}(X_{0})} \lambda^{+}(Z) d\mu \leq \theta \int_{\Gamma_{m_{0}}(X_{0})} \lambda^{+}(X_{0}) d\mu,$$

for all vector fields $Z$ which are $2\varepsilon_{0}$-C$^1$-close of $X_{0}$ and $Z = X_{0}$ outside $\Gamma_{m_{0}}(X_{0})$.

We observe that such $\varepsilon_{0}$ exists because $L(\cdot, \Gamma_{m_{0}}(X_{0}))$ is upper semicontinuous and $\Gamma_{m_{0}}(X_{0})$ is invariant, both for $Z_{0}$ and $Z^\varepsilon$. Recursively, knowing $X_{n-1}, m_{n-1}$ and $\varepsilon_{n-1} \in (0, 2^{-n})$, we define $X_{n} \in \mathcal{X}_{\mu}^{\infty}(M)$, $m_{n} \in \mathbb{N}$, and eventually $\varepsilon_{n} > 0$.

By Proposition 9.47, there exists $m_{n} \in \mathbb{N}$ and a perturbation of $X_{n-1}, X_{n} \in \mathcal{X}_{\mu}^{\infty}(M), \varepsilon_{n-1}$-C$^1$-close to $X_{n-1}$, with $X_{n} = X_{n-1}$ outside $\Gamma_{m_{n}}(X_{n-1})$ and such that

$$\int_{\Gamma_{m_{n}}(X_{n-1})} \lambda^{+}(X_{n}) d\mu < \delta_{n},$$

We assume that $m_{n} \geq m_{n-1}$ and note that $\Gamma_{m_{n}}(X_{n}) \subseteq \Gamma_{m_{n}}(X_{n-1}) \subseteq \Gamma_{m_{n-1}}(X_{n-1})$.

If $\int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(X_{n}) = 0$, then we finish the argument by taking $Y = X_{n}$, $D = A_{m_{n}}(Y)$ and $O$ a full measure subset of $\Gamma_{m_{n}}(Y)$. Otherwise, if $\int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(X_{n}) > 0$, we choose $\varepsilon_{n} \in (0, \varepsilon_{n-1}/2)$ such that $B(X_{n}, 2\varepsilon_{n}) \subseteq B(X_{n-1}, \varepsilon_{n-1})$ and also

$$\int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(Z) d\mu \leq \theta \int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(X_{n}) d\mu,$$

for all vector fields $Z$ which are $2\varepsilon_{n}$-C$^1$-close to $X_{n}$ and $Z = X_{n}$ outside $\Gamma_{m_{n}}(X_{n})$.

We continue this procedure recursively and if we obtain $\int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(X_{n}) d\mu = 0$ for some $n \in \mathbb{N}$, then we conclude the proof. Otherwise the sequence $\{X_{n}\}_{n \geq 0}$ converges in the C$^1$ topology to some $Y \in \mathcal{X}_{\mu}^{\infty}(M)$. Moreover, since $\varepsilon_{n} < \varepsilon/2^{n}$, we conclude that $Y$ is $\varepsilon$-C$^1$-close to $X$.

If we set $D = \bigcup_{n \geq 0} A_{m_{n}}(X_{n})$, since $A_{m_{n}}(X_{n}) \supseteq A_{m_{n-1}}(X_{n-1})$ and $Y = X_{n}$ at $A_{m_{n}}(X_{n})$, then $Y^\varepsilon$ has an $m_{n}$-dominated splitting at $A_{m_{n}}(X_{n})$.

Letting $Y := D_{\varepsilon}^\varepsilon = \bigcap_{n \geq 0} \Gamma_{m_{n}}(X_{n})$, then $Y \subseteq \Gamma_{m_{n}}(X_{n})$. To finish the proof of Theorem 9.4, we now check that $\int_{\Gamma} \lambda^{+}(Y) d\mu = 0$.

We note that $Y \subset B(X_{n}, 2\varepsilon_{n})$ for all $n \in \mathbb{N}$. Hence we have

$$\int_{\Gamma} \lambda^{+}(Y) d\mu < \int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(Y) d\mu \leq \theta \int_{\Gamma_{m_{n}}(X_{n})} \lambda^{+}(X_{n}) d\mu \leq \theta \delta_{n} \to 0.$$
We conclude that we have zero Lyapunov exponents in a full measure subset $O$ of $\mathcal{F}$. The proof of Theorem 9.4 is complete.

Now we consider the reason why Theorem 9.4 is stated for a dense subset instead of a residual subset. In [53], the authors developed a strategy to obtain a residual subset; unfortunately, this strategy does not apply in our case. Let us see why: they start with a $C^1$ system $X$ which is a continuity point of the entropy function. Then they define the "jump" (see [53, pp 1467]) of this semi-continuous function which is an integral over $\Gamma_m(X) := \bigcap_m \Gamma_m(X)$. Being a continuity point implies that the "jump" is zero. So, by definition of "jump", $\mu(\Gamma_m(X)) = 0$ or $\lambda^+(p) = 0$ for $\mu$-a.e. point $p \in \Gamma_m(X)$ and the statements of Theorem 9.4 are verified. In order to estimate a lower bound for the "jump" we perturb the original vector field $X$ as we did to prove Theorem 9.4. But our conservative flow-box theorem may not be applied to $X$, unless $X$ is of class $C^2$, and so this argument only works for $X \in \mathcal{X}_C^2(M)$. However this set, equipped with $C^1$ topology, is not a Baire space, and so in general residual sets are meaningless.

As explained at the beginning of Sect. 9.3, this can be extended to a full dichotomy for $C^1$ generic incompressible vector fields using the extension of the Bowen result for positive volume invariant sets having a dominated decomposition for the Linear Poincaré Flow.